SOLAR POWER – A GREAT MODELING STORY

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As mathematicians teaching undergraduate mathematics courses, we have an enormous responsibility to prepare our students to make important decisions that will shape the future of our planet and impact their wallets and an enormous opportunity to engage them and help them understand the importance of math and science.

This paper tells a modeling story and a story about a theorem – the Maximum Power Transfer Theorem. It is a story about how people use and sometimes misuse theorems. Books about electricity and electric circuits are filled with foundational ideas – for example, Kirchhoff's Laws and the Maximum Power Transfer Theorem – often associated with famous names and called "theorems" and "laws." But, theorems and laws can be misleading when used out of context. The heart of this story is the Modeling Cycle shown in Figure 1 and the interplay between theory and experimentation.

Because we can't see charges flowing through wires, we rely on modeling and analogy to build our understanding of electricity [1, 2]. We begin with a source of electricity – like a battery or a solar panel. We are interested in direct current and
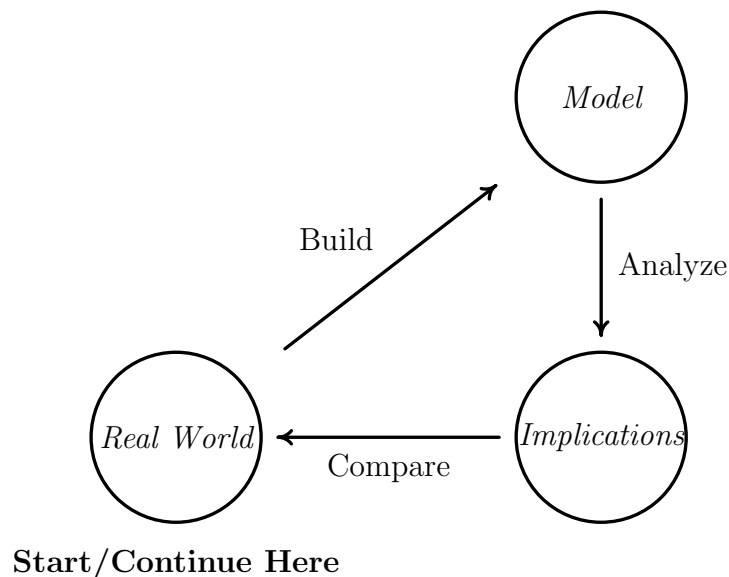
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Start/Continue Here

Figure 1: The modeling cycle

the following key concepts and their relationships: (1) Potential – measured in volts and often called "voltage;" (2) Current – measured in amperes; (3) Resistance – measured in ohms; (4) Power – measured in watts; and (5) Energy – measured in watt-hours, watt-seconds, or kilowatt-hours. Power is the derivative of energy.

The relationship between the current, i, flowing between two points in a circuit, the voltage, V, between the two points, and the resistance, R, between the two points is Ohm's Law: $V = iR$. This relationship is a great application when we study or review linear functions. It is also a good venue in which to introduce the equipment we will use later in this story. We use the following gear.

- A battery holder that holds two AA batteries.

- A rheostat or potentiometer that we use as a voltage divider. Together with the battery holder above this gives us a source whose voltage can varied from zero to roughly 3 volts. We bought this at a local Radio Shack. Ours is rated at 3 watts and 25 ohms.

- A fixed 10 ohm resistor rated at 20 watts. The exact resistance is unimportant

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but 10 ohms works well. Note that if the resistance is too low you may draw too much current for the current probe.

- A Vernier current probe. For this experiment the standard 600 milliamp probe[1] ($39) works fine but you may want to purchase the high current probe[2] ($79) that is needed for later experiments.

- A Vernier voltage probe. For this experiment the standard six volt probe[3] ($39) works fine but you may want to purchase the high voltage probe[4] ($49) that is needed for later experiments.

- A Vernier LabQuest 2.[5] ($329)

All of this equipment is very flexible and can be used for many different experiments. We have our students do the following simple experiment shown in Figure 2.

- Connect the rheostat across the battery holder with the two leads going to the two outer connectors on the rheostat.

- Connect one end of the fixed resistor to the middle connector on the rheostat.

- Connect the voltage probe across the fixed resistor and the current probe in series with the fixed resistor. Connect both probes to the LabQuest 2. Set the rheostat so that the voltage reading is at a minimum.

- Set the LabQuest 2 to record data at a rate of 20 times per second for thirty seconds.

- Start the data collection. Then slowly and steadily rotate the shaft on the rheostat until the voltage reading is at a maximum.

After data collection is complete, download the data using Vernier's free Logger Lite[6] software and export it to a spreadsheet. We use only the data collected while you

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[1]http://www.vernier.com/products/sensors/current-sensors/dcp-bta/
[2]http://www.vernier.com/products/sensors/current-sensors/hcs-bta/
[3]http://www.vernier.com/products/sensors/voltage-probes/dvp-bta/
[4]http://www.vernier.com/products/sensors/voltage-probes/30v-bta/
[5]http://www.vernier.com/products/interfaces/labq2/
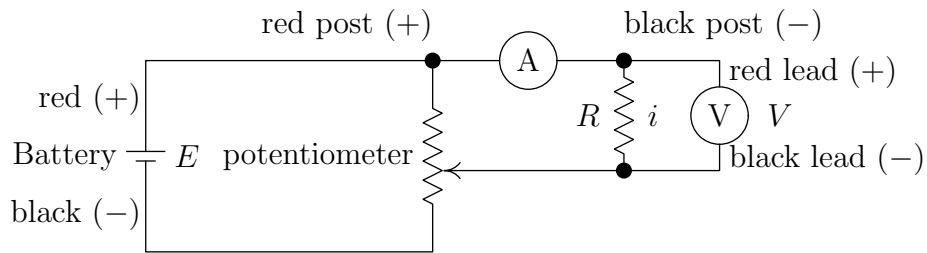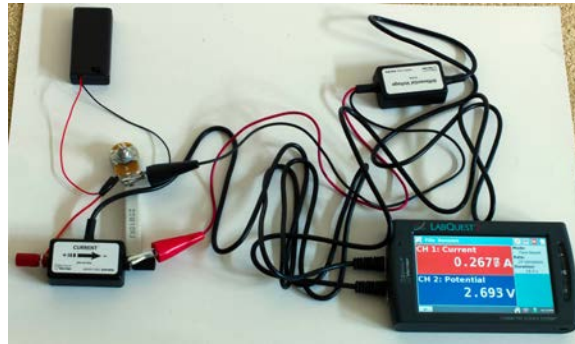[6]http://www.vernier.com/products/software/logger-lite/

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Figure 2: Experiment 1

were rotating the rheostat shaft – while the voltage supplied to the fixed resistor was changing. Create a scatter plot with current on the x-axis and voltage on the y-axis. You should see a line whose slope is close to the rated resistance of the fixed resistor.

We are now ready to move on to power. The power, P, used as current flows through a resistor is

$$P = iV = \frac{V^2}{R} = i^2 R.$$

The energy consumed from time a to b is

$$W = \int_a^b P \; dt.$$

We are interested in the power produced and the energy capacity of sources like batteries and solar panels. We begin with a simple circuit with an energy source, like a battery, and a "load," like a cell phone. The lower the resistance of the load,
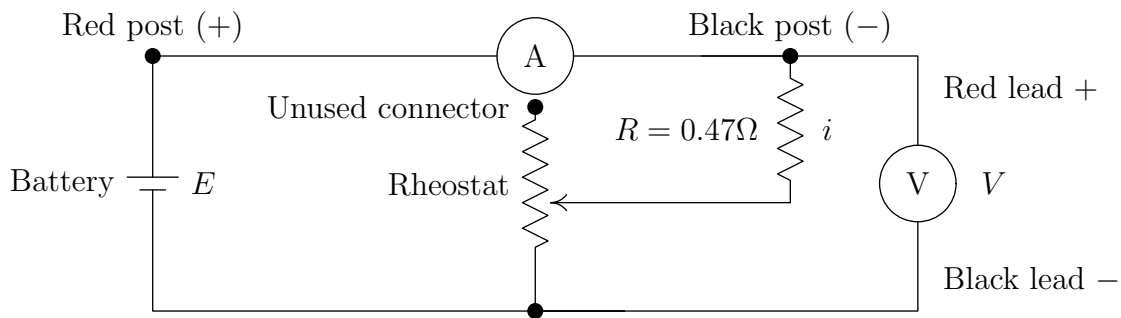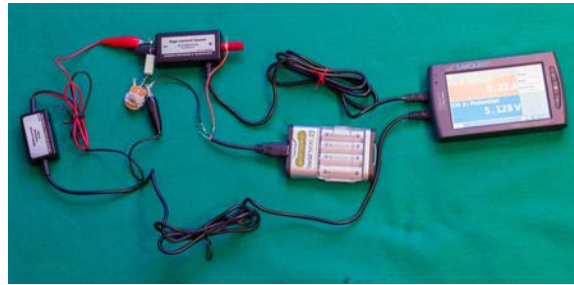
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Figure 3: Experiment 2

the more current we are trying to draw. If we ask for too much current, however, the source may not be able to keep up and the voltage will fall. We experience this, for example, on the hottest summer days when air conditioners are turned way up and we experience brown-outs.

The next experiment is shown in Figure 3. We use the rheostat to vary the resistance of a simulated load. We collect voltage and current data as we vary the resistance of the load.

We compute the resistance from voltage and current using the relationship $V = iR$ and we draw scatter plots of both voltage and power as functions of resistance. See Figures 4 and 5. With our apparatus it is hard to get good data for very low load resistance but you can see a hint – in Figure 5 the power does seem to fall off if the load resistance is too low. We are now ready to model the relationship between the resistance of the load and the power delivered by the battery.

Figure 6 shows a typical figure accompanying the Maximum Power Transfer Theorem. At the left we have an ideal source, perhaps a battery, whose voltage is E.
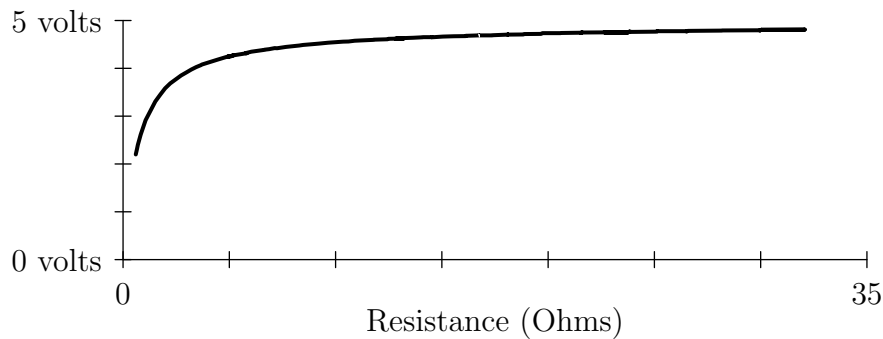
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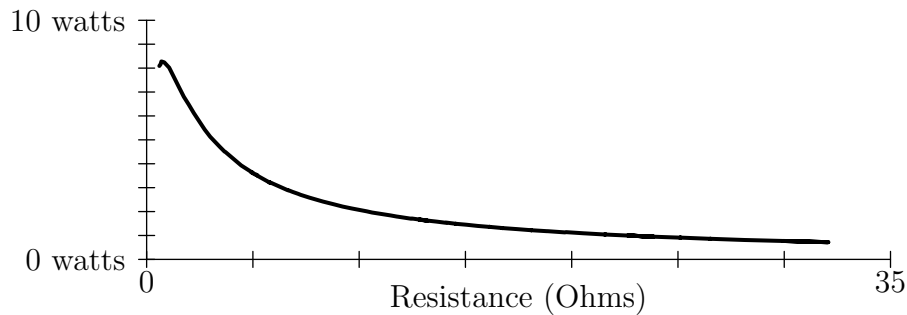Figure 4: Voltage as a function of resistance

Figure 5: Power as a function of resistance

The source is connected to a load whose resistance is labelled R. The load is at the right. A voltmeter is connected in parallel with the load and measures the voltage, V, across the load. When you use a battery in a circuit like this, as in our experiment, you can feel the battery heat up. Thus, the battery itself is an additional (but wasteful) load. In Figure 6 the load of the battery is shown separately from the battery but they are really in the same package – indicated by the dashed box in Figure 6. The resistance of this load is called the internal resistance of the battery and denoted R_{int}. We let i denote the current flowing through this circuit. Notice that $E = i(R_{int} + R)$. So

$$i = \frac{E}{R_{int} + R}.$$

This model focuses on two aspects of the source: (1) It is like a pump, pumping
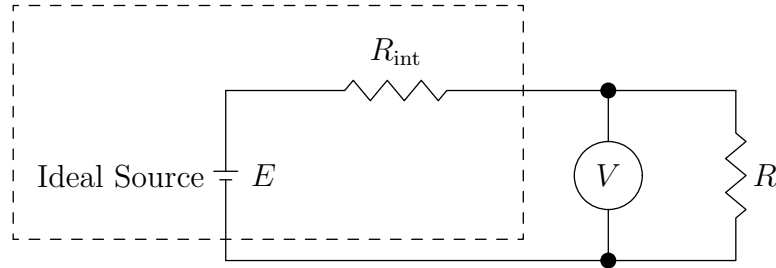
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Figure 6: An Illustration Accompanying the Maximum Power Transfer Theorem

charge "uphill." We measure how far uphill the pump pumps charge using the units, volts. We use the notation E for this "electromotive force." (2) It also resists the flow of charge through the battery itself. This resistance is measured in Ohms and denoted by R_{int}. In class we fit a model of the form

$$V = iR = \frac{ER}{R_{\text{int}} + R}.$$

to our data using E and R_{int} as parameters. A good ballpark estimate for E is 5 volts for four AA rechargeable batteries and a good initial estimate for R_{int} is 0.8 ohms. The model we get from this data is extremely good and the values we get for E and R_{int} are close to what is expected and very consistent across different experiments. Normally, at this point we'd include some graphs comparing the models to the data but they are so close that the graphs don't really show any differences.

Now, with this model in hand, we want to maximize the power consumed by the load,

$$P = iV = i(iR) = i^2 R = \left(\frac{E}{R_{\text{int}} + R}\right)^2 R.$$

This is a simple Calculus I problem. The maximum power is realized when $R = R_{\text{int}}$ – that is, when the load resistance is the same as the internal resistance of the battery.

Unfortunately, many people stop here but a little more analysis shows that the power consumed by the load is the same as the power consumed by the internal resistance of the battery. Since the power consumed by the load is $i^2 R$ and the
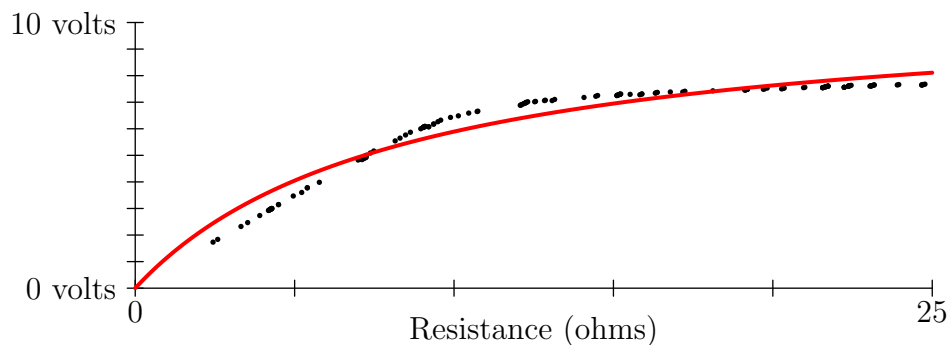
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Figure 7: Comparing our model and data for a solar panel

power wasted by the internal resistance of the battery is $i^2 R_{\text{int}}$, at maximum power the battery's efficiency is only 50%. We can realize higher efficiency if we are willing to sacrifice power by raising the resistance of the load. This is an important part of modeling – contending with competing measures of success.

Now we turn to solar panels. Naively applying the Maximum Power Transfer Theorem leads to the conclusion that we can realize the maximum possible power from a solar panel by using a load whose resistance is the same as the internal resistance of the solar panel. Figure 7 shows an attempt to fit the model

$$V = iR = \frac{ER}{R_{\text{int}} + R}$$

to data from one run of an experiment similar to the earlier experiment using a battery as a power source. For this experiment we used a solar panel made by GoalZero[7] with the high-current current probe and the high-voltage voltage probe from Vernier. The model fails dismally.

Figure 8 shows a graph of the useful power – that is, the power used by the load – as a function of the load resistance for the same data as shown in Figure 7. We focus on this relationship because we want to maximize the useful power generated by a solar panel. Notice that for this particular experiment maximum power is achieved with a load resistance near 10 ohms.

The total power produced by the solar panel including both the useful power

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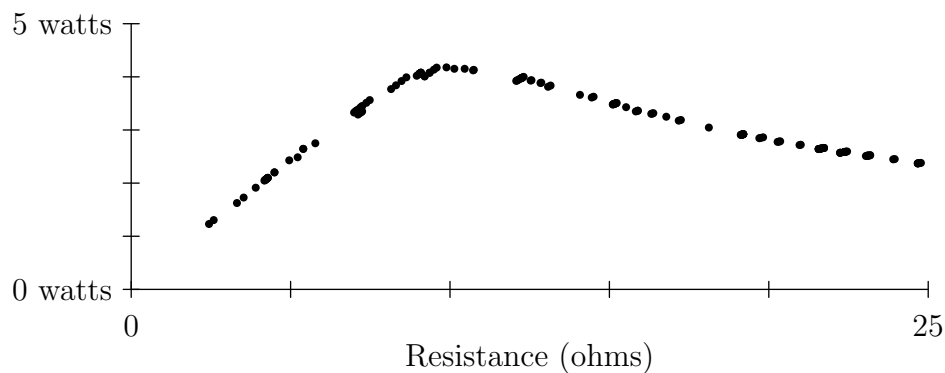[7]http://www.goalzero.com/shop/p/79/Guide-10-Plus-Solar-Kit/1:4/

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Figure 8: Data – useful power and load resistance – solar panel

through the load and the wasted power is

$$P = i^2(R + R_{\text{int}}).$$

This relationship is based on the assumption that the internal resistance R_{int} is a constant feature of the solar panel. Figure 9 shows a graph of this based on the data (black dots). If we assume that both the electromotive force E and the internal resistance R_{int} are constant features of the solar panel then

$$P = \frac{E^2}{R + R_{\text{int}}}.$$

This relationship is the red curve in Figure 9. Figure 10 is similar to Figure 9 except that it is based on our data and analysis for a battery.

The predictions of our model together with Figure 10 are very satisfying – the total power (wasted and useful) supplied by the battery decreases as the resistance of the load increases – and match our understanding that the rate of the chemical reaction inside the battery decreases as the load resistance increases.

In contrast, neither our model nor Figure 9 are at all satisfying. The data and the predictions of our model do not match. Neither one matches our expectation that the power received by the solar panel, which should all be turned into either useful or wasted energy, does not depend on the resistance of the load. For a battery the model based on constant electromotive force, E, and constant internal resistance,
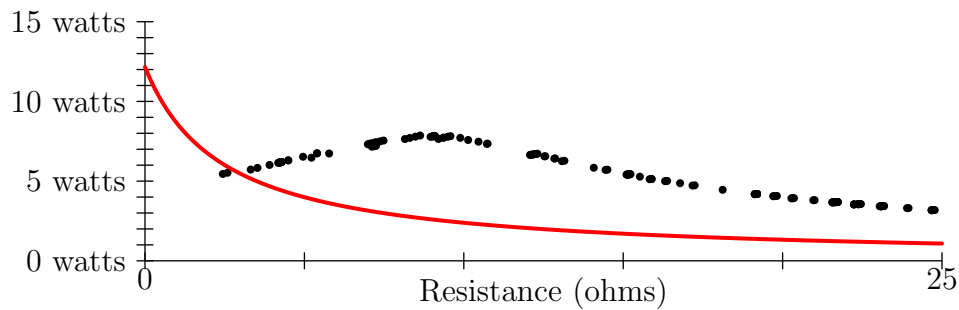
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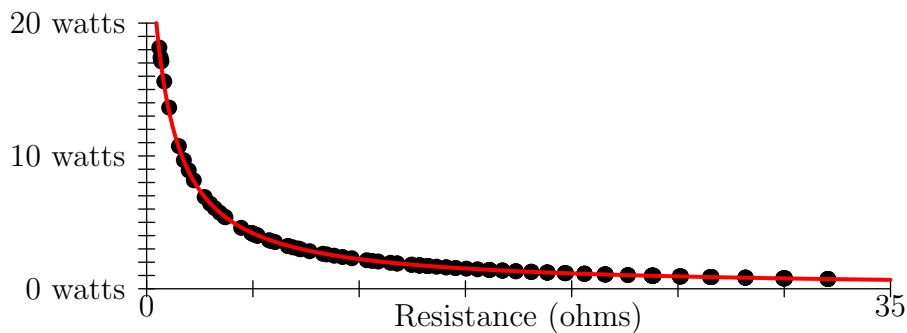Figure 9: Total power and load resistance – solar panel

Figure 10: Total power and load resistance – battery

R_{int}, is quite useful. For a solar panel the same model does not work at all. The power wasted by a solar panel might be accounted for in several ways but it is clear that it cannot be modeled by a constant internal resistance. We need new models for solar panels replacing the model that was so successful for analyzing the relationship between load resistance and power for batteries. This question sets us and our students up for additional experimentation and modeling.

References

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[2] Hayt, William H., Kemmerly, Jack E., and Durbin, Steven M. – *Engineering Circuit Analysis*, 7th Edition, 2006, McGraw-Hill Higher Education. ISBN-13:978-0072866117.

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