CLIMATE SCIENCE
WHY MATHEMATICIANS SHOULD BE INTERESTED

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Climate change and the climate science underpinning our understanding of climate change present us, as mathematicians and, especially as mathematicians teaching undergraduate mathematics classes, with both an enormous responsibility and an enormous opportunity. Our students will be making personal and public policy decisions that will affect our only world. They will be able to make sound decisions only if they have a real understanding of the Earth's climate system – including not only its physics, chemistry, and biology, but also the way in which our activity is an important driver of climate change.

In this paper we look at several examples showing how climate science can be woven into the undergraduate mathematics curriculum, helping our students develop an understanding of climate science at the same time they build their understanding of mathematics and how mathematicians think and work. Our study of the Earth's climate system is hampered by three fundamental problems.

- We have only one Earth (see Figure 1) and, thus, the gold standard of experimental science – comparing an experimental world and a control world is beyond our reach.
- Climate change occurs over long periods of time – beyond a class period, semester, or even a few years.
- Climate change occurs over vast geographical areas – beyond the confines of a laboratory or a campus.

For these reasons modeling and mathematical modeling play a huge role in building our understanding. Of equal importance, we cannot test possible alternative courses-of-action in the full real world. Thus, we must use models to test possible alternative courses-of-action. Climate change deniers often build their arguments around their skepticism about the value of models. Our students must be able to make their own informed decisions about the value of models.

One of the most well-known skeptics is Freeman Dyson. The following quote[1] shows his view about climate models.

> My first heresy says that all the fuss about global warming is grossly exaggerated. Here I am opposing the holy brotherhood of climate model experts and the crowd of deluded citizens who believe the numbers predicted by the computer models. Of course, they say, I have no degree in meteorology and I am therefore not qualified to speak. But I have studied the climate models and I know what they can do. The models solve the equations of fluid dynamics, and they do a very good job of describing the fluid motions of the atmosphere and the oceans. They do a very poor job of describing the clouds, the dust, the chemistry and the biology of fields and farms and forests. They do not begin to describe the real world that we live in. The real world is muddy and messy and full of things that we do not yet understand. It is much easier for a scientist to sit in an air-conditioned building and run computer models, than to put on winter clothes and measure what is really happening outside in the swamps and the clouds. That is why the climate model experts end up believing their own models.

[1] http://www.edge.org/3rd_culture/dysonf07/dysonf07_index.html Accessed 12 April 2010.

Figure 1: Our home

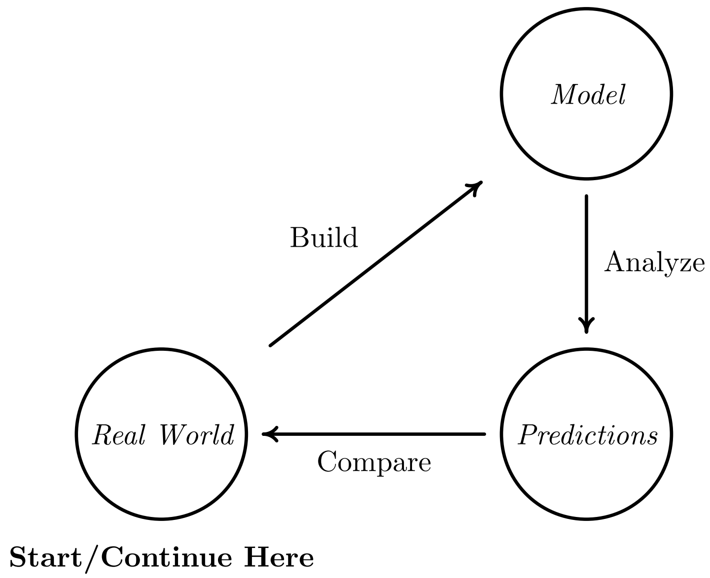

Figure 2: The modeling cycle

Dyson ignores the many climate scientists who do both modeling and field work. In fact, most of the climate scientists the authors know are much happier in Antarctica or Mongolia than in their offices. Most climate scientists love the outdoors and nature and were drawn into climate science in large part because of that love.

This paper emphasizes that, while many experiments are impossible, we still can use both observation and experiment to test our models. Figure 2 is the heart of this paper. We repeat the "modeling cycle" many times, building ever more useful models. Each turn of the modeling cycle begins in the real world with observation and experiment and each turn of the modeling cycle suggests additional observations and experiments. The experiments we discuss in this paper use inexpensive and readily available equipment and fit into the confines of the classroom.

1. A First Model

We begin with a simple example that uses only a bit of algebra and calculation. The first step in understanding the Earth's climate is building very simple models that can shed some insight into the Earth's temperature. See Figure 3. Our Earth receives energy in the form of light or electromagnetic radiation from the Sun. All objects whose temperature is above absolute zero, including the Earth, emit energy in the form of electromagnetic radiation in every direction. This phenomenon is called **black body radiation**. We see

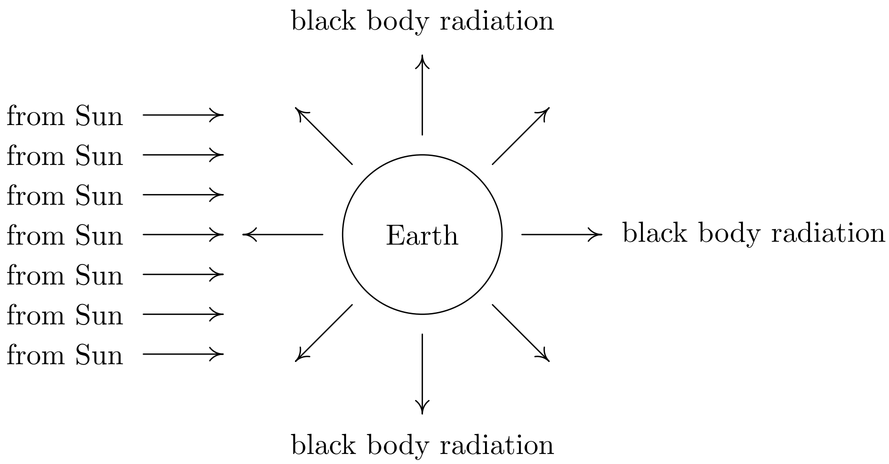

Figure 3: First climate model

black body radiation often – for example, the glowing wires in a space heater. As an object gets hotter the intensity of the black body radiation it emits rises and the color of the radiation shifts. This phenomenon can be used to measure an object's temperature. Figure 4 shows a widely available and inexpensive Black and Decker thermal leak detector that is based on this principle and is often used to find thermal leaks.

We begin by modeling the Earth by a sphere of radius R and we assume that it receives energy at a constant rate, Q, from the Sun. See Figure 3. This rate is approximately 1367.6 watts per meter^2. This means that a surface whose area was one square meter facing the Sun from a point on the Earth's orbit would receive 1367.6 watts, or joules per second. The rate Q is called the *energy flux density.* The value of Q actually varies slightly depending on whether the earth is at perihelion (closest to the Sun) in its orbit or at aphelion (furthest from the Sun). It also varies slightly depending on solar activity.

The area that the Earth presents to the Sun is a disk whose radius is the same radius as the radius of the Earth. We denote this radius by R_{earth}, measured in meters. Thus, the Earth receives

$$\text{incoming energy rate} = \pi R_{\text{earth}}^2 \times 1367.6 \text{ watts.}$$

As mentioned earlier, the rate at which an object whose temperature is above absolute zero emits black body radiation depends on its temperature. This rate is given by the Stefan-Boltzmann Law,

Figure 4: An inexpensive Black and Decker thermal leak detector

$$I = \sigma T^4,$$

where T is the temperature, measured using the Kelvin scale in units denoted K, called *kelvins.*

The temperature 0 K is absolute zero. Water freezes at a temperature of 273.15 K and boils at 373.15 K. The Kelvin scale is closely related to the Celsius scale. The magnitude of a degree in the Celsius scale is the same as the magnitude of a kelvin in the Kelvin scale but the zero point is different. For the Celsius scale the zero point is the temperature at which water freezes. For the Kelvin scale the zero point is absolute zero. An object whose temperature is 0 K has no thermal energy and emits no black body radiation. The constant σ is the Stefan-Boltzmann constant and has the value

$$\sigma = 5.67 \times 10^{-8} \text{ watts K}^{-4} \text{ per meter}^2.$$

Notice that I has units watts per meter^2.

Since the area of the Earth's surface is $4\pi R_{\text{earth}}^2$, the outgoing energy rate for the Earth is given by

$$\text{outgoing energy rate} = I 4\pi R_{\text{earth}}^2 = \sigma T^4 4\pi R_{\text{earth}}^2 \text{ watts.}$$

When the incoming energy rate is greater than the outgoing energy rate, the Earth's temperature will rise; when the incoming energy rate is lower than the outgoing energy rate the Earth's temperature will fall; and when the incoming and outgoing energy are equal then the Earth's temperature will remain constant. At this temperature we say the planet is in *thermal equilibrium*. We can find the thermal equilibrium by solving the equation

$$\begin{aligned}
\text{incoming energy} &= \text{outgoing energy} \\
\pi R_{\text{earth}}^2 \times 1367.6 &= \sigma T^4 4\pi R_{\text{earth}}^2 \\
T^4 &= \frac{1367.6}{4\sigma} \\
&= \frac{1367.6}{4 \times 5.67 \times 10^{-8}} \\
T &= \left(\frac{1367.6}{4 \times 5.67 \times 10^{-8}}\right)^{\frac{1}{4}} \approx 278.7 \text{ K.}
\end{aligned}$$

This value, about 5.5 degrees Celsius, is not in agreement with the known average temperature of the Earth, about 16 degrees Celsius. Not surprisingly, we need a better model. This was only our first turn around the modeling cycle. Comparing our model with the real world reveals the model's shortcomings and motivates a second turn around the modeling cycle.

This first model omitted a number of important factors. The first factor we want to add involves reflection – some of the incoming energy rate from the Sun is reflected back out into space. Snow, ice, and clouds, for example, reflect a great deal of the incoming light from the Sun. We use the term *albedo* to measure the total reflectivity of the Earth (or the Moon or another planet). Figure 5 shows cadets at the United States Military Academy investigating albedo by measuring the temperature of cars of different colors in a parking lot using an inexpensive Black and Decker thermal leak detector. The Earth's average albedo is about 0.3, which means that roughly 30% of the incoming solar radiation is reflected back out and roughly 70% of the incoming energy is absorbed by the Earth's surface. Thus, the net incoming energy rate for the Earth is

$$\text{incoming energy} = \left(\pi R_{\text{earth}}^2 \times 1367.6 \times 0.7\right) \text{ watts}$$

Figure 5: Using a Black and Decker Thermal Leak Detector to investigate albedo

and the equilibrium temperature is

$$T = \left(\frac{1367.6 \times 0.7}{4 \times 5.67 \times 10^{-8}} \right)^{\frac{1}{4}} \approx 254.9 \text{ K}.$$

Although this model is better, its prediction is worse than the prediction of our first model. Time for another turn around the modeling cycle. Before starting another turn, the following questions illustrate how we can use models to answer "what-if" questions.

Question 1 *One estimate for the variation in energy flux density due to the eleven year sunspot cycle is 1.3 watts per meter*^{2}. *What effect would a variation of this magnitude have on the Earth's temperature,* T*?*

Question 2 *What effect would an increase of 1% in the radius of the Earth's orbit have on the Earth's temperature,* T*?*

Figure 6 can shed some light on the shortcomings of our model. This figure shows data collected in a car parked in the open from 7:30 AM until 12:40 PM. The air temperature

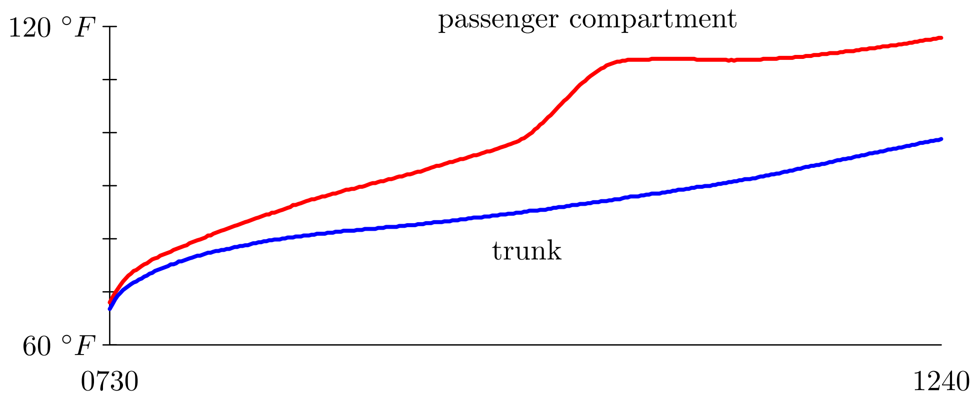

Figure 6: Temperature in the passenger compartment and trunk of a vehicle in the sun

was about 90°F at noon and the sky was partly cloudy in the morning. On this particular morning the temperature in the passenger compartment (red) rose to almost 120°F and in the trunk (blue) to just under 100°F.

The temperatures in the passenger compartment and the trunk of the car rise over the course of the morning for several reasons – for example, the sunlight warms the exterior of the car and some of that heat is transferred to the trunk and the passenger compartment. One big difference between the trunk and the passenger compartment is that the passenger compartment has windows. Sunlight enters the passenger compartment through the windows and heats up the interior surfaces. These surfaces radiate heat, just like the Earth, in the form of black body radiation. Although some of this radiation escapes through the windows, most of it is intercepted by the other interior surfaces of the car. As a result the passenger compartment of the car becomes much hotter than the trunk.

We did this experiment using inexpensive equipment, the LabQuest and two temperature probes, from Vernier Software and Technology.[2] The equipment we used uses rechargeable batteries that can be degraded by heat. In fact, the instruction manual warns against leaving the battery in a vehicle. We just did this experiment a couple of times.

The Stefan-Boltzmann Law states that the rate at which a body radiates energy in the form of electromagnetic radiation depends on its temperature. In addition, the frequency distribution of the radiation also depends on its temperature. The Sun is extremely hot and, as a result, most of its electromagnetic radiation is at higher frequencies – the frequencies we call visible light. Fortunately for us, the Earth is much cooler. Most of its black body radiation is at much lower frequencies – infrared frequencies. Water vapor, carbon dioxide,

[2] http://www.vernier.com

and other greenhouse gases are transparent to electromagnetic radiation at the frequencies produced by the Sun but trap electromagnetic radiation at the lower frequencies produced by the Earth. Because greenhouse gases, like carbon dioxide, trap some of the outgoing radiation, the Earth's equilibrium temperature is higher than the temperatures predicted by our first two models. Electromagnetic radiation from the Sun (mostly visible light) passes through the Earth's atmosphere, just like it passes through the windows of a car. Some of the black body radiation produced by the Earth is trapped by greenhouse gases in the atmosphere just like some of the black body radiation produced by the warm surfaces inside the car is trapped by the non-glass interior surfaces of the car.

2. Anthropogenic Climate Science 101

This section involves just a bit of calculation and shows the power of simple calculation. In the next section we use the same data and look at some examples that can be used in a statistics or data analysis class.

Figure 7 shows data collected by Charles Keeling and others on the concentrations of atmospheric carbon dioxide at Mauna Loa for over 50 years starting in 1957. The latest data set is always available at a web site[3] maintained by the Scripps Institution of Oceanography. A recent article[4], *A Scientist, His Work and a Climate Reckoning,* in the New York Times discusses Keeling and his work. This data set has played a large role in our understanding of global climate. Even though our atmosphere seems large, it is finite and the extra carbon dioxide produced by our combustion of fossil fuels appears to be having a measurable effect on the concentration of carbon dioxide in our atmosphere.

Since we started burning large amounts of fossil fuels the concentration of carbon dioxide in the atmosphere has risen dramatically. Because of the greenhouse effect this is expected to lead to an increase in the Earth's temperature – an increase that is already apparent. The climate change we are seeing is in large part anthropogenic – that is, caused by our own actions. The questions below lead to an easy plausibility check.

Question 3 *Estimate the total mass of the Earth's atmosphere. Hint: Air pressure at sea level is about 15 pounds per square inch. This is essentially the weight of the column of air above that square inch.*

This is a good place to talk about assumptions. As you answered the question above you undoubtedly made some assumptions to simplify the problem and your calculations. For example, you probably assumed that the Earth was a sphere. The Earth, however, is not a sphere. Most of the Earth's surface is covered by oceans and is at sea level but we also

[3] http://scrippsco2.ucsd.edu

[4] http://www.nytimes.com/2010/12/22/science/earth/22carbon.html?_r=1

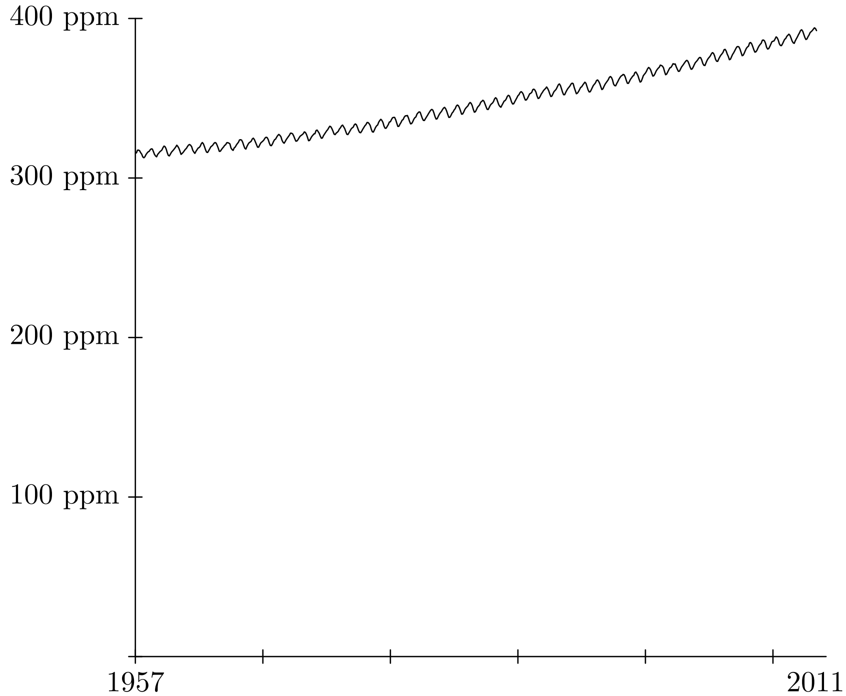

Figure 7: Keeling data from Mauna Loa

Year	Million Tonnes	Year	Million Tonnes	Year	Million Tonnes
1981	3836.1	1991	4557.1	2001	4917.9
1982	3980.0	1992	4519.2	2002	4960.8
1983	3986.5	1993	4395.8	2003	5313.8
1984	4191.0	1994	4484.1	2004	5723.1
1985	4441.0	1995	4605.4	2005	6049.0
1986	4548.7	1996	4680.2	2006	6356.5
1987	4649.7	1997	4730.4	2007	6588.3
1988	4755.2	1998	4651.9	2008	6822.1
1989	4837.9	1999	4638.2	2009	6904.6
1990	4740.3	2000	4701.4	2010	7254.6
				2011	7695.4

Table 1: Coal production in millions of (metric) tonnes

have some high mountains and the air pressure at higher elevations is much lower than the air pressure at sea level.

Question 4 *Does the assumption that the Earth is a sphere cause your estimate of the total mass of the atmosphere to be an overestimate or an underestimate? Estimate by how much your estimate might be off due to this assumption.*

The analysis in the preceding question is an example of *sensitivity analysis* and is an important part of modeling. Models are essentially simplifications based on simplifying assumptions and it is important to consider the effects of these simplifications.

Now that we know how large (in mass) the atmosphere is we need to look at how much carbon dioxide we have added to the atmosphere and whether the increase we see in the atmospheric concentration of carbon dioxide could be due to this source. The 2012 BP Statistical Report on World Energy,[5] is a good source of data on energy consumption. The data in Table 1 comes from this source.

Question 5 *Based on the data in Table 1 estimate the mass of the carbon dioxide, CO_2, produced by burning coal over the past 31 years. Assume that coal is 100% carbon. Since the molecular weight of carbon dioxide is 44 and the atomic weight of carbon is 12, carbon dioxide is only 12/44 carbon so that burning one ton of coal produces 3.7 tons of carbon dioxide. Note that a metric ton (tonne) is 1000 kilograms, or 2200 pounds.*

[5] http://www.bp.com/sectionbodycopy.do?categoryId=7500&contentId=7068481
http://www.bp.com/extendedsectiongenericarticle.do?categoryId=9041233&contentId=7075263 accessed 25 July 2012.

Based on your calculation do you think is it possible that burning fossil fuels is responsible for a substantial part of the increase in the atmospheric concentration of carbon dioxide? Recall that we are only looking at one fossil fuel and only at the last 31 years.

Figure 7 shows an increase in atmospheric carbon dioxide but not a steady increase. There is a seasonal oscillation that is due to plant and animal activity. During the spring and early summer plant activity is at a maximum and the process of photosynthesis converts carbon dioxide to oxygen and in the process brings the atmospheric concentration of carbon dioxide down. During the fall and winter photosynthesis activity decreases and metabolism and decay predominate. These processes convert oxygen into carbon dioxide and cause the atmospheric concentration of carbon dioxide to rise. Globally the northern hemisphere has more boreal forests and for this reason the seasonal variation in atmospheric carbon dioxide is linked to the seasons in the northern hemisphere.

Question 6 *What effect do you think changing land use patterns (more parking lots and clearing forests and jungles) might have on the atmospheric concentration of carbon dioxide?*

3. Data Analysis and Descriptive Modeling

We begin by looking at one of the most common and overused methods of modeling. Our message is primarily but not entirely cautionary. We use the Keeling data shown in Figure 7 as an example. This data involves two variables – time (measured in years) and the atmospheric concentration of carbon dioxide (measured in parts per million). We use the notation $(t_1, y_1), (t_2, y_2), \dots (t_n, y_n)$, where $n = 641$, for the data points, with t_i denoting the time of the i^{th} data point and y_i denoting the atmospheric concentration of carbon dioxide for the i^{th} data point.

The goal of descriptive modeling is to find a function $f(t)$ that enables us to predict the atmospheric concentration of carbon dioxide on the basis of time. The word "predict" is the problem – it implies that we will be able to use the function $f(t)$ to predict the future – that is, to extrapolate beyond the time period for which data is available. We will see that this is simply untrue. This kind of descriptive modeling is often good for interpolating – filling in holes where there is missing data – during the time period for which we have information when that information is incomplete. Using this kind if modeling for prediction, however, can be more problematic. We will look at the usual kinds of descriptive models people build.

Figure 8 shows our first attempt – a linear model. This model is

$$y = 1.45295t - 2536.64$$

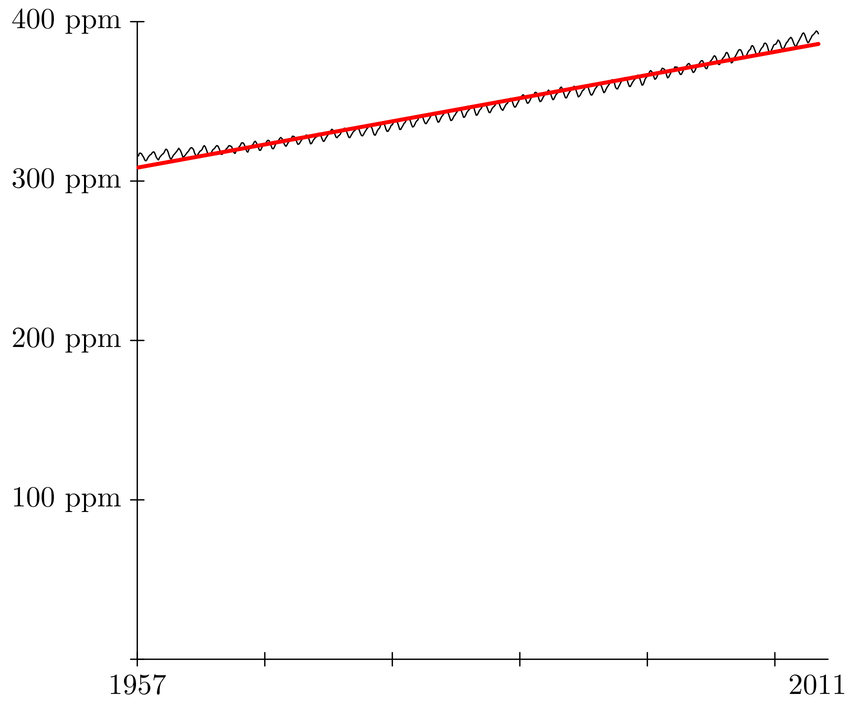

Figure 8: A linear model for the Keeling data from Mauna Loa

and has a coefficient of determination of 0.9774.

Figure 9 shows our second attempt – a quadratic model. This model is

$$y = 0.0123323t^2 - 47.5034t + 46,046.5$$

and has a coefficient of determination of 0.9908. This model seems to capture the general trend very well but does not capture the seasonal oscillation.

Now we want to try to capture the seasonal oscillation by fitting a function of the form

$$y = at^2 + bt + c + p\sin(2\pi t) + q\cos(2\pi t)$$

to the data. Figure 10 shows the result – the function

$$y = 0.0121743t^2 - 46.8763t + 45,424.2 + 2.60296\sin(2\pi t) - 1.03\cos(2\pi t)$$

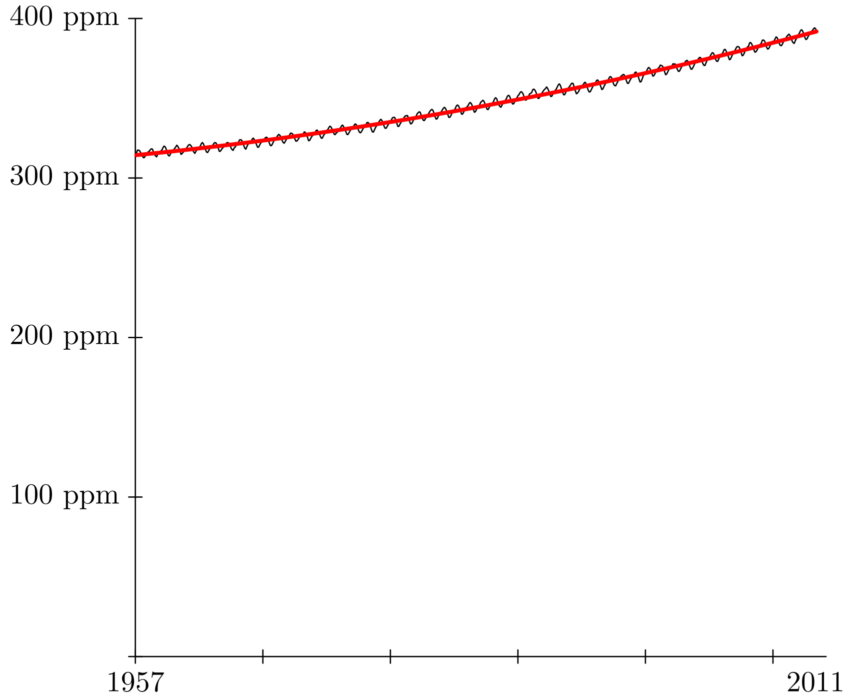

Figure 9: A quadratic model for the Keeling data from Mauna Loa

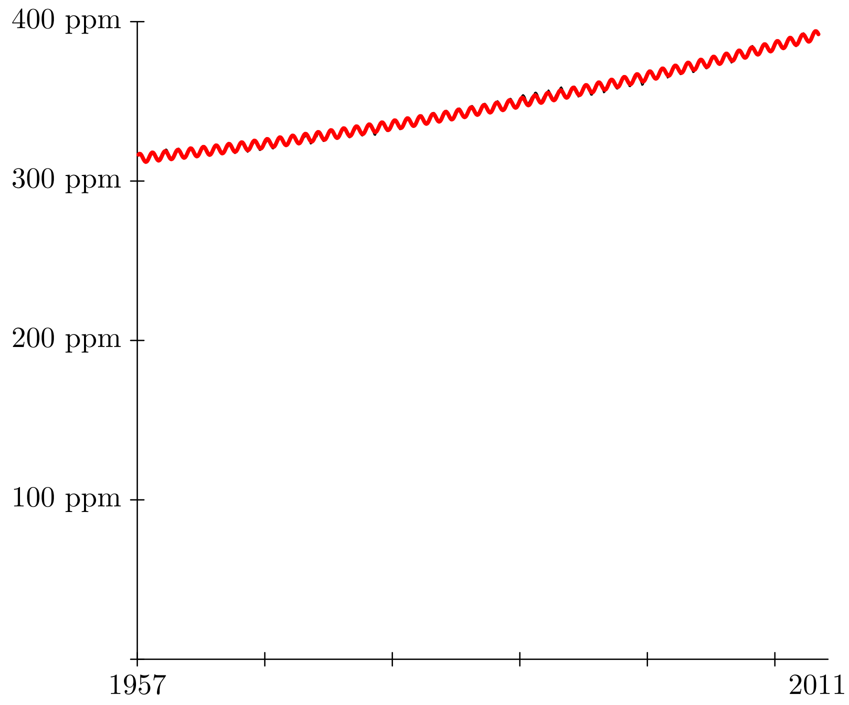

Figure 10: A seasonal model for the Keeling data from Mauna Loa

with a coefficient of determination of 0.9984. This model looks extremely good – it would bring joy to the heart of many modelers.

We, however, are more careful. The usual purpose of this kind of modeling is making predictions – let's test this model by using it to make predictions and seeing how it fares. We could wait ten or fifteen years but that is too long to wait, if we want to use our model now. Instead we build a model using only the first 400 of our 641 data points and see how well this model predicts the remaining data points. The model we get is

$$y = 0.0188258t^2 - 73.1164t + 71,303.4 + 2.49074\sin(2\pi t) - 1.08481\cos(2\pi t).$$

This model has a coefficient of determination of 0.9969 – it looks very good based on this coefficient of determination. Figure 11, however, reveals that it does not do very well at predicting the last 241 data points. As we mentioned earlier – models like this should be used very cautiously, if at all, for predicting the future.

The next series of questions look at the same data set but a different series of models.

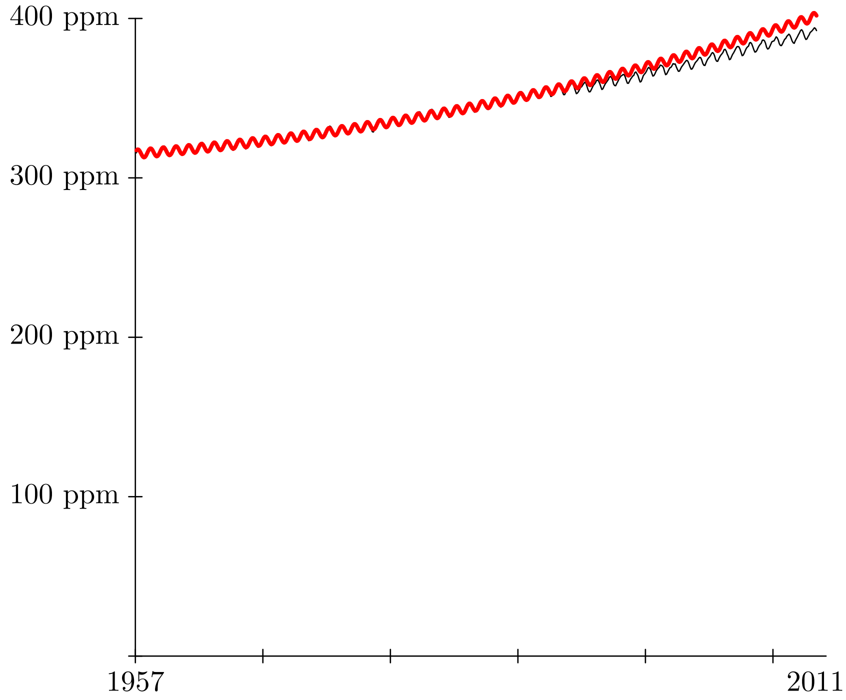

Figure 11: A model that was built using only the first 400 data points

Question 7 *Build an exponential model for the full Keeling data set. Use the form*

$$y = ab^{(t-1958)} + c$$

to avoid some numerical issues that might arise with the form

$$y = ab^t + c.$$

Question 8 *Build an exponential model with seasonal oscillations for the full Keeling data set. Use the form*

$$y = ab^{(t-1958)} + c + p\sin(2\pi t) + q\cos(2\pi t).$$

Question 9 *Build an exponential model with seasonal oscillations using only the first 400 points from Keeling data set. Use the form*

$$y = ab^{(t-1958)} + c + p\sin(2\pi t) + q\cos(2\pi t).$$

See how well this model does at predicting the remaining 241 data points.

4. Surprises – Feedback and a Link Between the Hydrosphere and Atmosphere

This section involves only division.

Keeling's data has a pronounced annual cycle that appears to be correlated with the seasons in the Northern Hemisphere. The carbon dioxide concentration declines during the period of greatest photosynthetic activity in the Northern hemisphere and rises during the period of lesser photosynthetioc activity. Although Mauna Loa is located in the Northern Hemisphere (roughly 21° N), the more significant factor appears to be the much higher concentration of boreal forests in the Northern Hemisphere. See Figure 12 from ecoworld.com.[6]

Because photosynthesis consumes carbon dioxide and produces oxygen, it would be expected to remove carbon dioxide from the atmosphere and add oxygen. Metabolism has the reverse effects. Because metabolism consumes oxygen and produces carbon dioxide, it would be expected to remove oxygen from the atmosphere and add carbon dioxide. During

[6] http://www.ecoworld.com/maps/world-ecoregions.html accessed February 17, 2011.

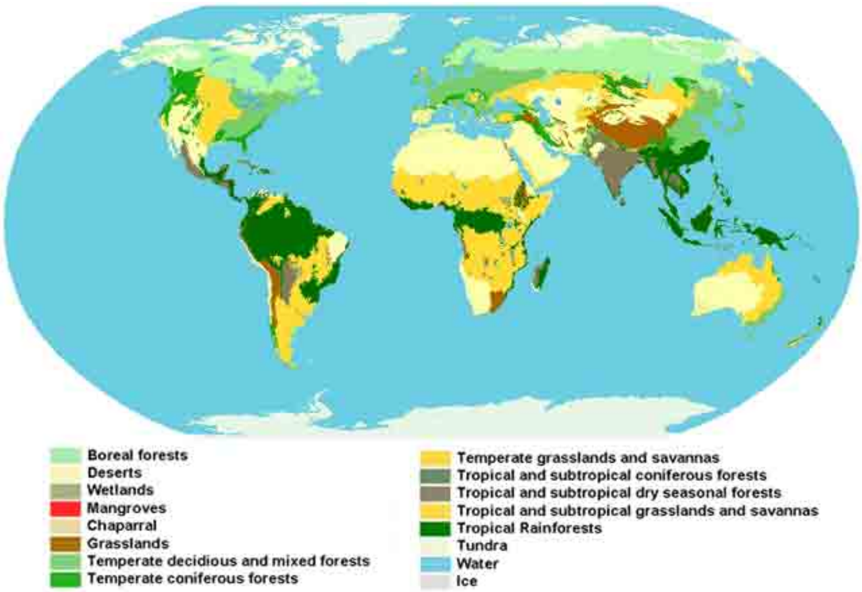

Figure 12: World map of ecoregions

the growing season photosynthesis predominates and as a result we would expect the concentration of oxygen in the atmosphere to rise and the concentration of carbon dioxide to fall. Outside the growing season we would expect the reverse – carbon dioxide would rise and oxygen fall. In this section we look at an experiment designed to test this hypothesis using a simulated daily cycle of night-and-day.

Figure 13 shows our first running experiment. We used an AeroGrow Aerogarden to simulate a 24 hour cycle with 16 hours of light and 8 hours of darkness. We placed leaves in a sealed chamber with sensors measuring the concentrations of carbon dioxide (in parts per million) and of oxygen (as a percent) in the chamber, a sensor outside the chamber measuring the light, and a LabQuest recording the measurements. With the exception of the AeroGrow, all of this equipment was purchased from Vernier Software and Technology.[7] Figure 14 shows a rough graph of what we expected.

- The concentrations of oxygen and carbon dioxide would rise-and-fall oppositely to each other.

- Oxygen would rise during the simulated day and fall during the simulated night.

[7] http://www.vernier.com

Figure 13: Our experimental apparatus

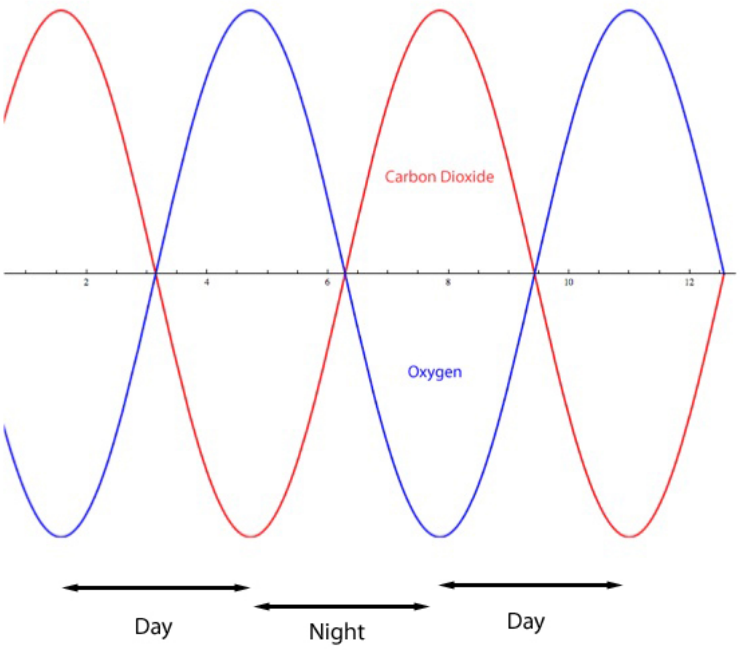

Figure 14: What we expected

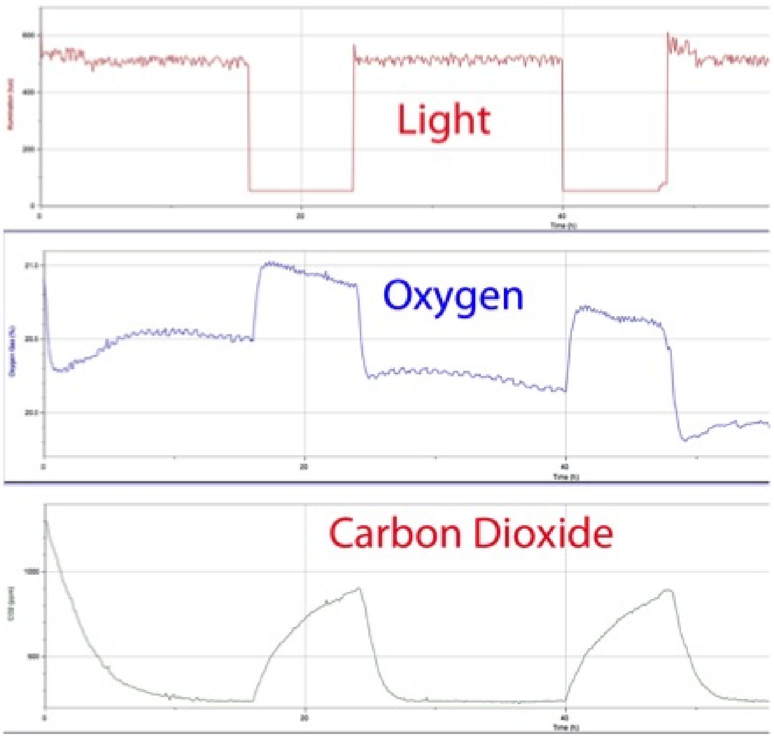

Figure 15: What we saw

- Carbon dioxide would fall during the simulated day and rise during the simulated night.

Figure 15 shows what we saw. Notice that

- When the light went off, the oxygen level rose suddenly and unexpectedly. The carbon dioxide level also rose as expected.
- During the night, after its initial sudden jump, the oxygen level fell as expected and the carbon dioxide level rose as expected.
- When the light came on, the oxygen level suddenly and unexpectedly fell. The carbon dioxide level also fell as expected.
- During the simulated day after the initial sudden jumps the carbon dioxide and oxygen levels behaved as expected until the carbon dioxide level was quite low.

After this initial surprise we came up with a reasonable hypothesis for what happened. There are three key ideas and we need some fourth-grade mathematics.

- The sensors do *not* measure the *amount* of either carbon dioxide or of oxygen. They do measure *concentrations* – either

$$\frac{O_2}{N_2 + O_2 + H_2O + \text{other}}$$

  for oxygen or

$$\frac{CO_2}{N_2 + O_2 + H_2O + \text{other}}$$

  for carbon dioxide.

- Quotients go up when denominators go up and down when denominators go down.
- Water is more soluble in warm air than in cold air.

With all this in mind consider what might happen when the light goes off.

- The temperature drops and some of the water vapor in the air condenses.
- The denominator falls.
- This causes the unexpected rise in the concentration of oxygen. The concentration of carbon dioxide also rises but we expected it to rise.

Now consider what happens when the light goes on.

- The temperature rises and water is drawn into the air as water vapor.
- The denominator rises.
- This causes the unexpected drop in the concentration of oxygen. The concentration of carbon dioxide also drops but we expected it to drop.

We tested this hypothesis by running a series of experiments. Figure 16 shows the results of one such experiment. The set up was similar to the first experiment but the chamber was empty except for some moist towels and we recorded relative humidity and temperature in

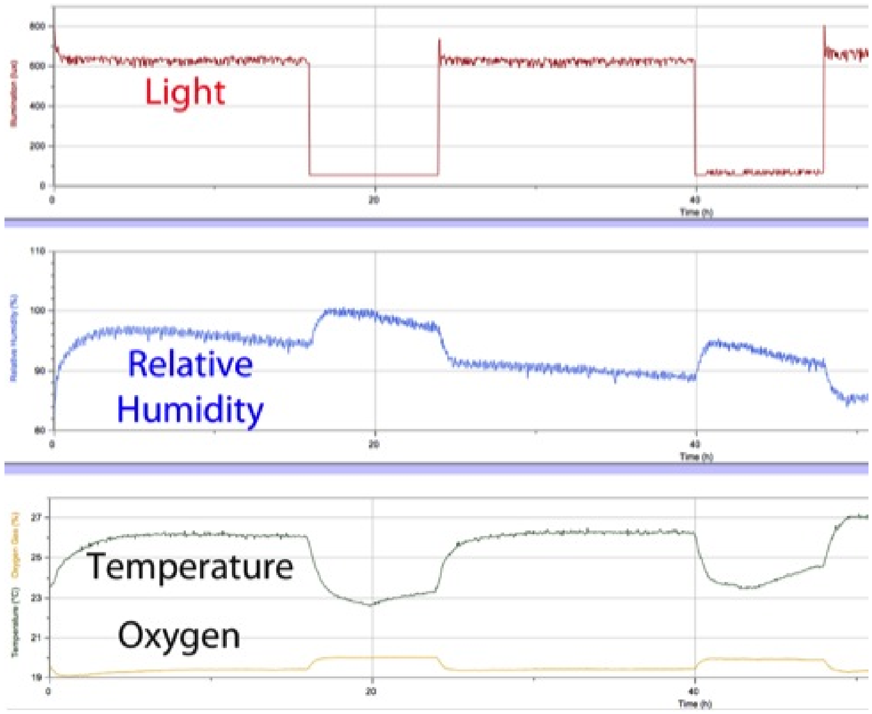

Figure 16: A second experiment

addition to light and oxygen. We did not record carbon dioxide. Notice the results support our hypothesis. Although there is a complicated relationship among temperature, relative humidity, and absolute humidity, the relative humidity graph supports the conjecture that something is going on with water vapor in the air. We also took time lapse photographs of the experiment and could see water condensing on the walls of the chamber when the light went out.

An experiment motivated by the interaction between the biosphere and the atmosphere forced us to confront the interaction between the hydrosphere and the atmosphere. This also brings up the idea of feedback. As the temperature of the atmosphere rises, water vapor is drawn into the atmosphere. Since water is a greenhouse gas this causes the temperature to rise further – a great example of positive feedback.

5. Climate Science 201: More Feedback, Latitude and Ice Ages

This section is more mathematically demanding. It is great material for a course in linear algebra or matrices. It also requires a good understanding of our three dimensional world.

Figure 17: A globe

This material can help students understand the seasons. The *Private Universe* video[8] and the project[9] from which it comes show how many people, even Harvard graduates, have a basic misunderstanding of the seasons. People who don't understand the seasons may very well have difficulty understanding climate science.

Now we add two elements to our model – the seasons and latitude. The best way to work through this section is by referring to a globe like the one shown in Figure 17. In addition, you may want to use the (free) simulations[10] produced by the **DIYModeling** project. [11] We measure latitude in the usual way. Points on the equator have latitude zero. The North Pole is at latitude 90 N and the South Pole is at latitude 90 S. The Earth revolves around an axis going through the North and South Poles once each day.

A glance at Figure 18 confirms what you already know – latitude makes a huge difference. We all know that it is colder at the poles than at the equator. Figure 18 shows a map of the Earth's albedo at different places. The albedo is highest at the poles, primarily because

[8] http://www.youtube.com/watch?v=p0wk4qG2mIg
[9] http://www.learner.org/teacherslab/pup/
[10] http://diymodeling.appstate.edu/node/29
[11] http://diymodeling.appstate.edu

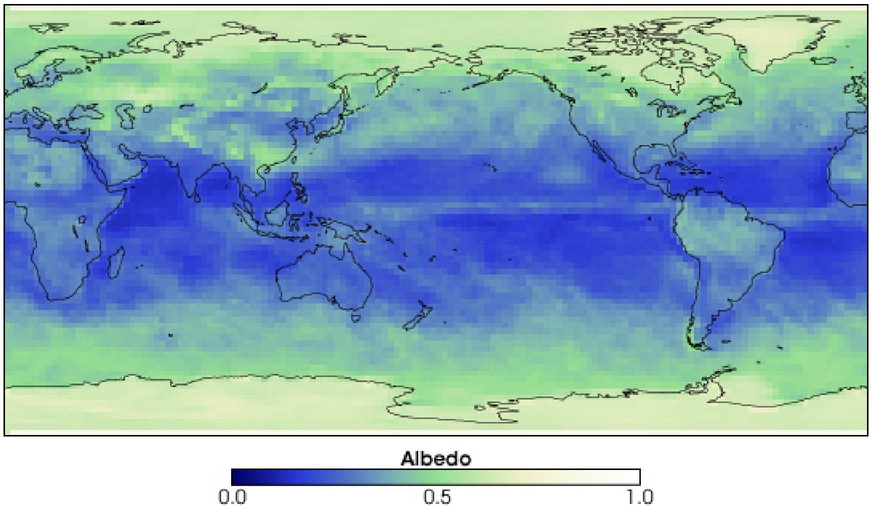

Figure 18: The Earth's albedo, March 2005, measured from NASA's Terra satellite

of ice, than at the equator. Figure 19 shows the basic reason why it is colder at the poles – the rate at which a particular point on the Earth receives energy per square meter from the Sun depends on the angle of the incoming Sun's rays. When the Sun is directly overhead the rate of energy received per square meter is much higher than when the Sun is "low in the sky." Two vectors are important – the vector $\vec{N}$, called the normal vector, that is perpendicular to the Earth's surface at the point of interest and the vector $\vec{S}$ that points from the point of interest toward the Sun. If I_0 denotes the rate at which energy is received per square meter when the Sun is directly overhead then the rate at which energy is received per square meter when the Sun is not necessarily directly overhead is

$$I = I_0 \cos\theta.$$

If both $\vec{S}$ and $\vec{N}$ are unit vectors then this formula becomes

$$I = I_0(\vec{N} \cdot \vec{S}).$$

If the Sun is below the horizon then $\vec{N} \cdot \vec{S}$ would be negative, so this formula must be changed to

$$I = I_0 \max(0, \vec{N} \cdot \vec{S}).$$

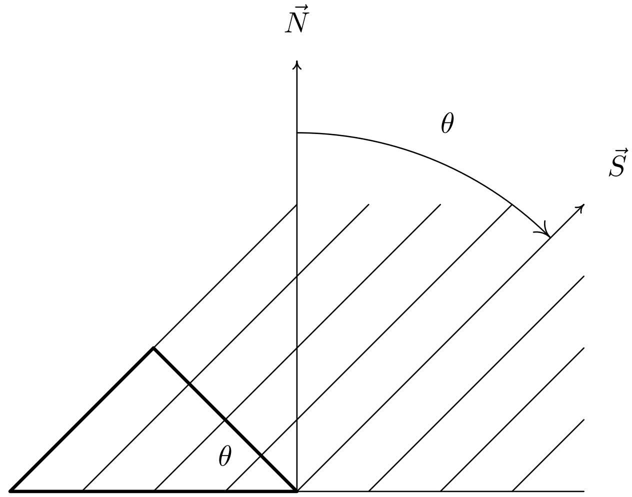

Figure 19: The Sun's angle and energy received

We need to study the way that the rate at which Earth receives energy from the Sun varies by latitude, time of day, and time of year. There are many possible sets of coordinates we might use. For our first coordinate system we put the origin at the center of the Earth and the z-axis running through the North and South Poles with the positive z-direction being north. The Earth will revolve around this axis and the x- and y-axes will remain fixed with the equator in the xy-plane and the positive x-axis going from the center of the Earth through the place where the prime meridian crosses the equator. See Figure 20. The point where the prime meridian crosses the equator is marked by a dot. The x-axis goes through this point. Notice that so far we are looking only at the Earth and this first coordinate system is attached to the Earth and revolves with the Earth.

We will use the letter ϕ to denote the latitude of a point, measured in radians. Thus, points for which $\phi = 0$ are on the equator, $\phi = \pi/2$ is the North Pole, and $\phi = -\pi/2$ is the South pole. Thus, in this coordinate system a point on the prime meridian whose latitude is ϕ has coordinates $\vec{u} = R_e\langle \cos\phi, 0, \sin\phi\rangle$.

Now we need a second coordinate system. This second coordinate system is shown in Figure 21. Its origin is at the center of the Earth and its z-axis goes through the two poles with the positive z-axis in the direction of the North Pole. The x- and y-axes, however, are fixed in space and do not revolve as the Earth revolves. If $\vec{u}$ denotes a position on the Earth in the first coordinate system and $\vec{v}(t)$ denotes the same point in the second coordinate system at time t measured in hours then

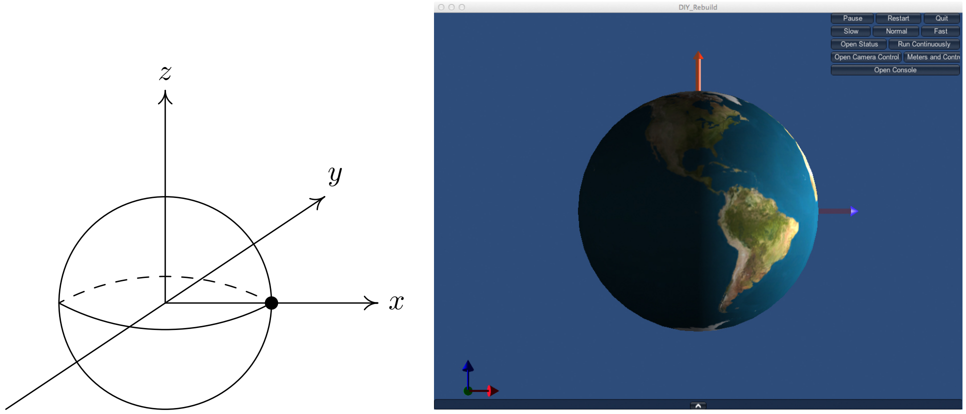

Figure 20: Revolving coordinate system for the Earth

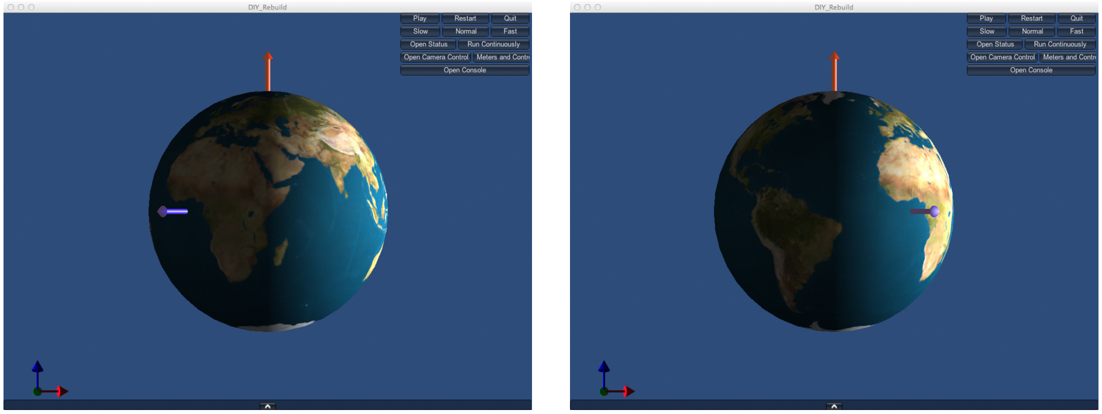

Figure 21: Two frames in a simulation in a fixed coordinate system for the Earth

$$\vec{v}(t) = \begin{bmatrix} \cos\left(\frac{2\pi t}{24}\right) & -\sin\left(\frac{2\pi t}{24}\right) & 0 \\ \sin\left(\frac{2\pi t}{24}\right) & \cos\left(\frac{2\pi t}{24}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{u}.$$

Notice that in the second coordinate system points on the Earth are rotating around the Earth's axis once every 24 hours. In this coordinate system over the course of a day a point $\vec{u} = R_e\langle \cos\phi, 0 \sin\phi \rangle$ on the prime meridian at latitude ϕ traces out a circle

$$\vec{v}(t) = \begin{bmatrix} \cos\left(\frac{2\pi t}{24}\right) & \sin\left(\frac{2\pi t}{24}\right) & 0 \\ -\sin\left(\frac{2\pi t}{24}\right) & \cos\left(\frac{2\pi t}{24}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_e \cos\phi \\ 0 \\ R_e \sin\phi \end{bmatrix} = \begin{bmatrix} R_e \cos\left(\frac{2\pi t}{24}\right)\cos\phi \\ -R_e \sin\left(\frac{2\pi t}{24}\right)\cos\phi \\ R_e \sin\phi \end{bmatrix}$$

or

$$\vec{v}(t) = R_e\left\langle \cos\left(\frac{2\pi t}{24}\right)\cos\phi, -\sin\left(\frac{2\pi t}{24}\right)\cos\phi, \sin\phi \right\rangle,$$

where R_e denotes the radius of the Earth, over the course of a day. The upward pointing unit normal vector at this point is

$$\vec{N} = \left\langle \cos\left(\frac{2\pi t}{24}\right)\cos\phi, -\sin\left(\frac{2\pi t}{24}\right)\cos\phi, \sin\phi \right\rangle.$$

Now we need a third coordinate system. This coordinate system is shown in Figure 22 and has two similarities to the second one. Its origin is at the center of the Earth and it is fixed in space. It does not revolve as the Earth revolves. The xy-plane, however, is not the plane of the equator. Instead it is the plane of the ecliptic – that is, the plane containing the Earth's orbit around the Sun. The angle between these two planes is 23.4°, or 0.408 radians. If $\vec{v}(t)$ represents a point on the Earth in the second coordinate system then this same point is represented in the third coordinate system by the vector $\vec{w}(t)$ given by

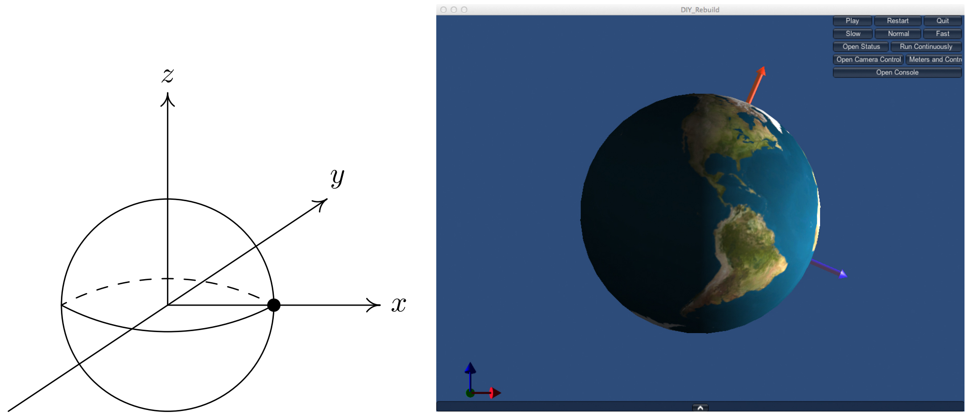

Figure 22: Tilting the Earth

$$\vec{w}(t) = \begin{bmatrix} \sin(0.408) & 0 & \cos(0.408) \\ 0 & 1 & 0 \\ -\cos(0.408) & 0 & \sin(0.408) \end{bmatrix} \vec{v}(t).$$

In this coordinate system the Sun appears to rotate about the Earth. If we measure time in years, then its position at time s is given by the vector

$$150,000,000\ \langle \cos(2\pi s), \sin(2\pi s), 0\rangle,$$

where we approximate the Earth's orbit around the Sun by a circle of radius 150,000,000 km. The unit vector pointing in this direction is

$$\vec{S} = \langle \cos(2\pi s), \sin(2\pi s), 0\rangle.$$

At a given time of day, a given latitude, and a given time of the year, we can put this all together to compute the rate at which a point on the prime meridian is receiving energy from the Sun per square meter using

$$I = I_0 \max(0, \vec{N} \cdot \vec{S}) = 1367.6 \max(0, \vec{N} \cdot \vec{S})$$

where the normal vector at the point $\vec{w}(t)$ is given by

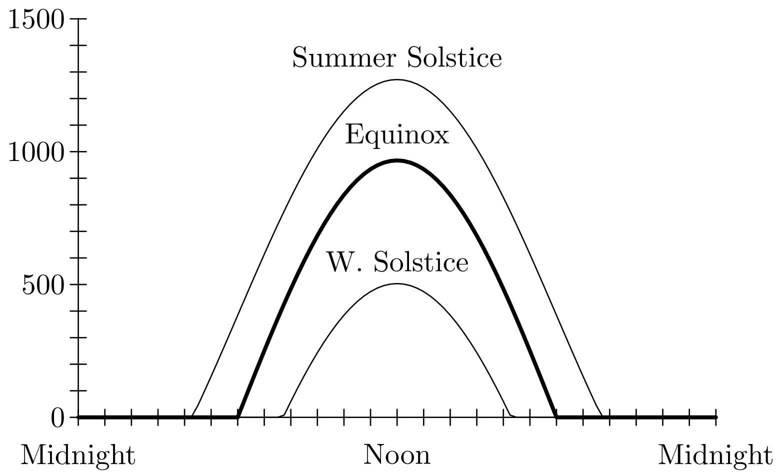

Figure 23: W/meter^2 at latitude 45 on the day of equinox and the two solstices

$$\vec{N} = \frac{1}{|\vec{w}(t)|}\vec{w}(t).$$

Figure 23 shows a graph of this rate at latitude 45 on the day of the summer solstice, the winter solstice and either equinox. Figure 24 shows the total energy received per square meter over the course of each day at three latitudes – the equator, 45 N, and the North Pole – at different times of the year. Figure 25 shows the total energy received per square meter over the course of a year as a function of latitude. Table 2 gives the annual average insolation in watts per square meter by latitude.

When the Earth is colder, more ice forms and this changes the Earth's albedo. Hans Kaper[12] models the dependence of albedo on temperature using the function

$$\alpha(T) = 0.7 - 0.4\left(\frac{e^{\frac{T-265}{5}}}{1+e^{\frac{T-265}{5}}}\right),$$

where T is temperature, for albedo. See Figure 26. The algebraic description of this function is not crucial. What is crucial is its general shape. At very low temperatures the albedo is 0.7 because of all the snow and ice. At very high temperatures the albedo is 0.3 because the snow and ice have melted. Close to the temperature at which water freezes we see a continuous change of the albedo.

[12] *Mathematics and Climate* (Draft).

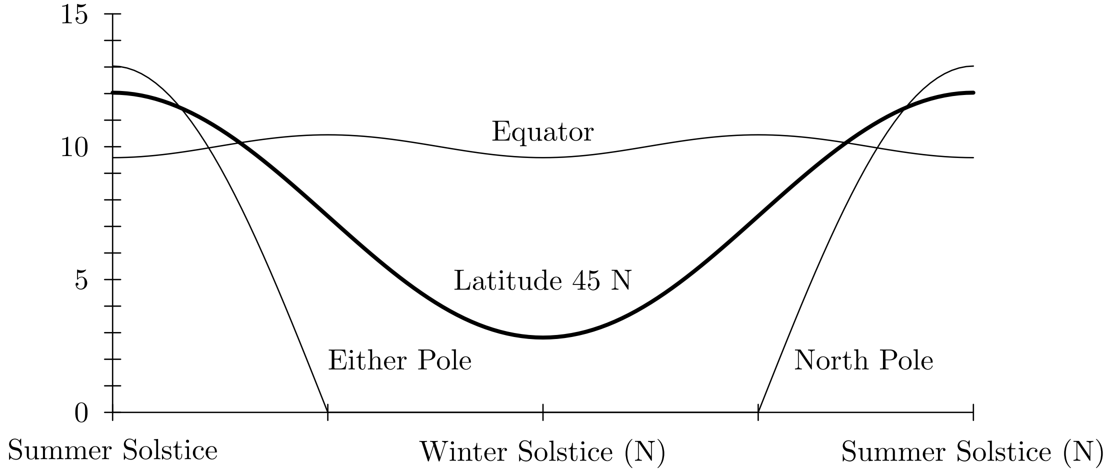

Figure 24: KWH/meter^2 per day at the equator, latitude 45, and a pole

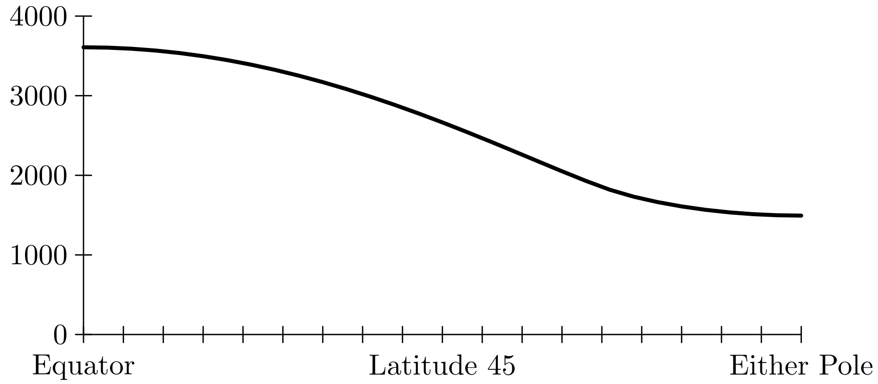

Figure 25: KWH/meter^2 per year as a function of latitude

Latitude	Insolation	Latitude	Insolation
equator	417.61	48	294.76
3	417.09	51	280.70
6	415.52	54	266.32
9	412.92	57	251.80
12	409.29	60	237.36
15	404.65	63	223.36
18	399.01	66	210.47
21	392.39	69	200.35
24	384.83	72	192.50
27	376.34	75	186.24
30	366.96	78	181.31
33	356.74	81	177.57
36	345.72	84	174.95
39	333.94	87	173.40
42	321.47	poles	172.88
45	308.39		

Table 2: Annual average insolation per square meter by latitude

This brings up another example of feedback – as the temperature drops the albedo will drop, which in turn causes the temperature to drop further. This kind of feedback is called *positive* because it magnifies any change.

Using this function for albedo, our formula for the incoming energy rate taking into account albedo,

$$\text{incoming energy rate} = \left(\pi R_{\text{earth}}^2 \times 1367.6 \times 0.7\right) \text{ watts,}$$

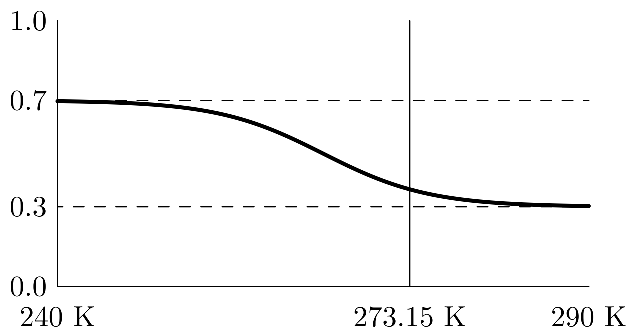

Figure 26: Albedo as a function of temperature (K)

is replaced by

$$\text{incoming energy rate} = \left(\pi R_{\text{earth}}^2 \times 1367.6 \times (1-\alpha(T))\right) \text{ watts}$$

Next, as a very rough model including the greenhouse effect, we replace our original function for the outgoing energy rate

$$\text{outgoing energy rate} = \sigma T^4 4\pi R_{\text{earth}}^2 \text{ watts}$$

by the function

$$\text{outgoing energy rate} = \epsilon\sigma T^4 4\pi R_{\text{earth}}^2 \text{ watts}$$

where ϵ is as constant between zero and one that specifies the fraction of the blackbody radiation that is not trapped by greenhouse gases. A good working value for ϵ is 0.60. This is the value we will use.

Question 10 *Investigate the equilibrium temperature of the Earth using the new incoming energy rate function and the new outgoing energy rate function with* $\epsilon = 0.6$.

So far we've worked with the average temperature of the entire Earth. Now we want to look at different latitudes. We begin by dividing both the total incoming energy rate and the total outgoing energy rate by the surface area ($4\pi R_{\text{earth}}^2$) of the Earth to obtain the incoming and outgoing energy densities in watts per square meter.

$$\begin{aligned}\text{incoming energy density} &= \frac{\left(\pi R_{\text{earth}}^2 \times 1367.6 \times (1-\alpha(T))\right) \text{ watts}}{4\pi R_{\text{earth}}^2} \\ &= \left(\frac{1367.6}{4}\right)(1-\alpha(T)) \text{ watts per meter}^2\end{aligned}$$

$$\begin{aligned}\text{outgoing energy density} &= \frac{\epsilon\sigma T^4 4\pi R_{\text{earth}}^2 \text{ watts}}{4\pi R_{\text{earth}}^2} \\ &= \epsilon\sigma T^4 \text{ watts per meter}^2\end{aligned}$$

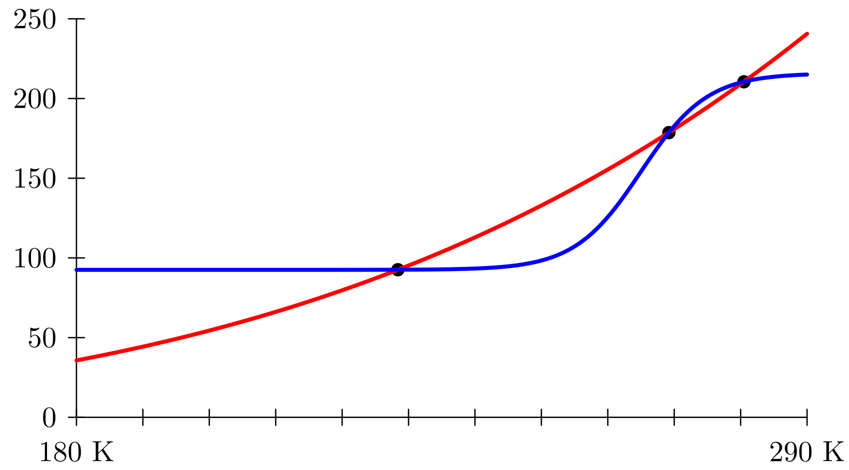

Figure 27: Incoming (blue) and outgoing (red) energy density at latitude 45

You may wonder about the factor

$$\left(\frac{1367.6}{4}\right)$$

in the expression for the incoming energy density. Recall that the energy flux from the Sun at our distance from the Sun is 1367.6 watts per square meter. This is an average over the entire Earth. At any given time only half of the Earth is illuminated by the Sun. It is night for the other half. So, just to get started, that 1367.6 would be divided by 2 giving us 1367.6/2. That figure would apply if the Sun was directly overhead of every point all day at every latitude but it isn't. If you take into account the angle of the Sun at each point you get the figure 1367.6/4 that we see in the expression for the incoming energy density.

We are interested in the annual average incoming energy density at each latitude – the numbers given in Table 2 – for example, the annual average incoming energy density at latitude 45 (N or S) is

$$308.39\,(1 - \alpha(T)).$$

Figure 27 shows the annual average incoming and outgoing energy densities at latitude 45. The curve for the incoming annual average energy is blue and the curve for the average annual outgoing energy is red. The three dots mark three equilibrium temperatures – the temperatures where the incoming and outgoing energy densities are equal.

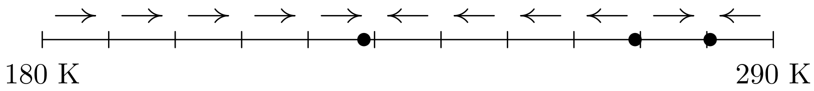

Figure 28: Phase diagram for temperature at latitude 45

Notice that

- To the left of the leftmost equilibrium temperature the outgoing energy density is less than the incoming energy density. Thus, if the temperature was below the leftmost equilibrium temperature we would expect it to rise toward the leftmost equilibrium temperature.
- Between the leftmost equilibrium temperature and the middle equilibrium temperature the outgoing energy density is greater then the incoming energy density. Thus, if the temperature was between these two equilibrium temperatures we would expect the temperature to fall toward the first equilibrium temperature.
- Between the middle equilibrium temperature and the rightmost equilibrium temperature the outgoing energy density is below the incoming equilibrium energy density. Thus, if the temperature was between these two equilibrium temperatures we would expect it to rise toward the rightmost equilibrium temperature.
- To the right of the rightmost equilibrium temperature the outgoing energy density is greater than the incoming energy density. Thus, if the temperature was above the rightmost equilibrium temperature we would expect it to fall toward the rightmost equilibrium temperature.

This information can be visualized using a *phase diagram* like the one shown in Figure 28. Points on the line represent values of the variable T or temperature. The dots represent the equilibrium points. The arrows represent the direction in which the variable T is changing.

- To the left of the leftmost equilibrium temperature the outgoing energy density is less than the incoming energy density. Thus, if the temperature was below the leftmost equilibrium temperature we would expect it to rise toward the leftmost equilibrium temperature. The arrows are pointing to the right, the increasing direction for T.
- Between the leftmost equilibrium temperature and the middle equilibrium temperature the outgoing energy density is greater then the incoming energy density. Thus, if the temperature was between these two equilibrium temperatures we would expect

the temperature to fall toward the first equilibrium temperature. The arrows are pointing to the left, the decreasing direction for T.

- Between the middle equilibrium temperature and the rightmost equilibrium temperature the outgoing energy density is below the incoming equilibrium energy density. Thus, if the temperature was between these two equilibrium temperatures we would expect it to rise toward the rightmost equilibrium temperature. The arrows are pointing to the right, the increasing direction for T.

- To the right of the rightmost equilibrium temperature the outgoing energy density is greater than the incoming energy density. Thus, if the temperature was above the rightmost equilibrium temperature we would expect it to fall toward the rightmost equilibrium temperature. The arrows are pointing to the left, the decreasing direction for T.

The three equilibrium points have different characters.

- The leftmost equilibrium point is an example of an *attracting* or *stable* equilibrium point. If the temperature is close to this equilibrium point it will be pulled in or attracted to it. This property motivates the adjective "attracting." If the Earth was at this equilibrium point and a slight perturbation pushed it off then it would be pulled back. This property motivates the adjective "stable."

- The middle equilibrium point is an example of a *repelling* or *unstable* equilibrium point. If the temperature is close to but not right on this equilibrium point it will be pushed away. This property motivates the adjective "repelling." If the Earth was exactly at this equilibrium point it would stay there but if the slightest perturbation pushed the Earth away from this equilibrium point it would move away to one of the other equilibrium points. This property motivates the adjective "unstable."

- The rightmost equilibrium point is an example of an *attracting* or *stable* equilibrium point. If the temperature is close to this equilibrium point it will be pulled in or attracted to it. This property motivates the adjective "attracting." If the Earth was at this equilibrium point and a slight perturbation pushed it off then it would be pulled back. This property motivates the adjective "stable."

This analysis shows why we can have ice ages and periods of mild temperatures. At latitude 45 there are two stable equilibrium points – one corresponds to an ice age and the other to a mild climate.

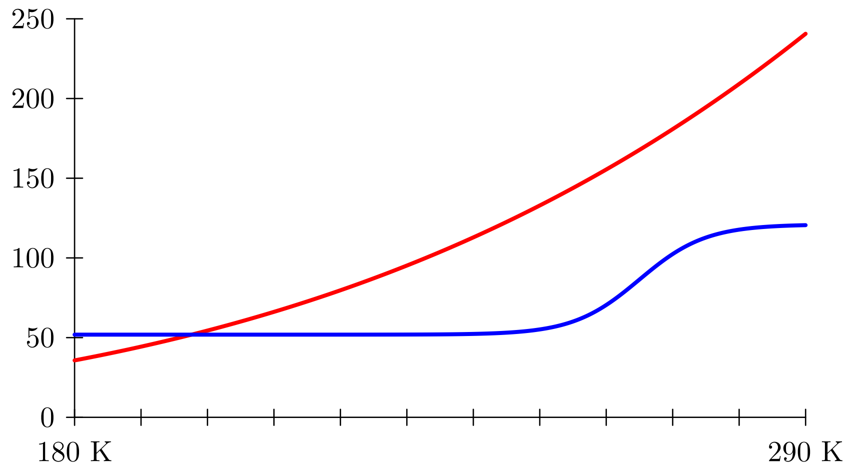

Figure 29: Incoming (blue) and outgoing (red) energy density at the poles

Question 11 *Figure 29 shows the incoming and outgoing energy densities at the poles. Using this figure draw a phase diagram for the temperature at the poles. Find and classify (attracting or repelling) the equilibrium point(s).*

Question 12 *Using your favorite graphing program and Table 2 draw the incoming and outgoing energy densities at the equator. Using this figure draw a phase diagram for the temperature at the equator. Find and classify (attracting or repelling) the equilibrium point(s).*