

USING SATELLITE ORBITS AND SPACE TRAVEL WITH
GAME-QUALITY SIMULATIONS IN MATH AND PHYSICS CLASSES
FROM HIGH SCHOOL THROUGH COLLEGE

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Figure 1: Satellite Laboratory

Figure 1 is a screen shot from a **DIYModeling** satellite laboratory in which we can explore the orbits of manmade satellites, like the International Space Station (ISS), or of natural satellites, like the Moon. This paper develops the underlying physics and mathematics and gives several examples of how this laboratory can be used in math and science courses in high school and college. Together these examples tell an important story that leads up to the study of Hohmann transfer and an introduction to interplanetary space travel and exploration. The **DIYModeling** software and all the simulations in this paper are available at the **DIYModeling** web site.¹

Traditionally students check their work by looking at answers in the back-of-the-book or, more recently, by getting feedback from a computer-based homework system. Students working on many of the examples discussed in this paper check their answers in simulations

¹Home – <http://diymodeling.appstate.edu> or ICTCM – <http://diymodeling.appstate.edu/node/66>.

– for example, a student trying to put a satellite into a geo-stationary orbit can check his or her answer by looking back toward the Earth from the satellite to see if the satellite remains in position. This has several advantages.

- Students are engaged by seeing whether their answers work.
- Students are more confident in their answers when they are right.
- Students understand when their answers are wrong. Observing a simulation often helps them understand what went wrong.
- Students are much more likely to persevere until they get a problem right.

1. Gravity

The magnitude of the gravitational force exerted by an object of mass M on another object of mass m is given by the expression

$$F = \frac{GMm}{r^2}$$

where G is the gravitational constant, 6.67384×10^{-11} meters³ kilogram⁻¹ second⁻² and r is the distance between the two objects. Since we are interested in a satellite of mass m orbiting around a much more massive object, like the Earth or the Sun, M is usually much larger than m . We will use the center-of-mass of the larger object as the origin. In reality the two objects move about their common center-of-mass but because M is so much larger than m we will assume that the larger object remains fixed at the origin.

The force exerted by the object of mass M on the object of mass m is pulling the object of mass m toward it. If \vec{p} denotes the position of the object of mass m then the force vector is given by

$$\vec{F} = \underbrace{\left(\frac{GMm}{\|\vec{p}\|^2}\right)}_{\text{magnitude}} \underbrace{\left(-\frac{1}{\|\vec{p}\|}\right)}_{\text{direction}} \vec{p} = -\left(\frac{GMm}{\|\vec{p}\|^3}\right) \vec{p}$$

and its acceleration by

$$\vec{p}'' = \frac{1}{m} \vec{F} = -\left(\frac{GM}{\|\vec{p}\|^3}\right) \vec{p}.$$

Because the product GM occurs so often, it is convenient to replace it by a single constant $\mu = GM$. Thus, our formulas above become

$$F = \frac{\mu m}{r^2}$$

$$\vec{F} = -\left(\frac{\mu m}{\|\vec{p}\|^3}\right)\vec{p}$$

$$\vec{p}'' = -\left(\frac{\mu}{\|\vec{p}\|^3}\right)\vec{p}.$$

The mass of the Earth is $M = 5.9736 \times 10^{24}$ kilograms. So for the Earth

$$\begin{aligned}\mu &= (6.67384 \times 10^{-11} \text{ meters}^3 \text{ kilogram}^{-1} \text{ second}^{-2})(5.9736 \times 10^{24} \text{ kilograms}) \\ &= 3.98669 \times 10^{14} \text{ meters}^3 \text{ second}^{-2}\end{aligned}$$

We will approximate the Earth by a sphere of radius 6.371×10^6 meters. Thus, at the surface of the Earth the calculated acceleration due to gravity is

$$\text{calculated } g = \frac{3.98669 \times 10^{14}}{(6.371 \times 10^6)^2} = 9.822 \text{ meters per second}^2.$$

Note that this figure does not agree with the commonly used value, $g = 9.80$ meters per second². What's wrong? Answering this question is a nice exercise and leads to our next topic – centrifugal force.

2. Centrifugal Force

The terms “centrifugal force,” “centripetal force,” and “centripetal acceleration” are often used loosely. One of our goals here is to build a solid understanding of these ideas. We begin, however, with a simple formula for “centripetal acceleration” found in Wikipedia²,

$$a = \frac{v^2}{r}$$

²http://en.wikipedia.org/wiki/Centrifugal_force Accessed 12 July 2012.

This formula applies to an object traveling at a constant speed s on a circular path of radius r . Consider, for example, an object at a constant height close to the Earth's surface at latitude 42 degrees. This object is making one revolution around a circular path whose radius is approximately $6,371,000 \cos(42\pi/180)$ meters every 24 hours. This gives it a speed of 344.31 meters per second and, using the formula above, its centripetal acceleration is 0.0186 meters per second². This much downward acceleration is needed just to keep the object at a constant height. So the acceleration we observe is the calculated acceleration above, 9.822 meters per second², minus this figure. This is close to the commonly used value, $g = 9.80$ meters per second².

3. Determining Centripetal Acceleration

Finding the formula

$$a = \frac{s^2}{r}$$

given in Wikipedia for centripetal acceleration is an easy set of exercises in calculus. The position of an object traveling at a constant speed around a circular path of radius r centered at the origin can be described by

$$\vec{p} = r\langle \cos ct, \sin ct \rangle$$

Quick calculations yield

$$\vec{p}' = cr\langle -\sin ct, \cos ct \rangle$$

and

$$\vec{p}'' = c^2r\langle -\cos ct, -\sin ct \rangle = -c^2\vec{p}.$$

Notice that the speed is $s = cr$ and the acceleration is toward the origin and has magnitude

$$c^2r = \frac{s^2}{r},$$

which is the formula from Wikipedia.

4. Orbital Speed, Period, and Geostationary Orbits

In the previous section we saw that the magnitude of the centripetal acceleration for an object in a circular orbit of radius r is related to its speed by

$$\text{magnitude of centripetal acceleration} = \frac{s^2}{r}$$

and that it is directed toward the center of the circular orbit.

Earlier we saw that the acceleration produced by gravity is given by

$$\left(\frac{GM}{\|\vec{p}\|^2} \right) \left(-\frac{1}{\|\vec{p}\|} (\vec{p}) \right) = \left(\frac{\mu}{\|\vec{p}\|^2} \right) \left(-\frac{1}{\|\vec{p}\|} (\vec{p}) \right)$$

For an object in a circular orbit around the origin this is directed toward the center of the orbit and has magnitude

$$\left(\frac{\mu}{\|\vec{p}\|^2} \right)$$

Thus, for a satellite in a circular orbit of radius r

$$\frac{s^2}{r} = \frac{\mu}{r^2}$$

and

$$s = \sqrt{\frac{\mu}{r}}.$$

The period is

$$\frac{2\pi r}{s} = \left(\frac{2\pi}{\sqrt{\mu}} \right) r^{3/2}$$

As an interesting exercise students can fit a curve of the form

$$\left(\frac{2\pi}{\sqrt{\mu}} \right) r^p$$

Planet	Perihelion(km)	Aphelion(km)	Period (years)
Mercury	46,001,200	68,816,900	0.240846
Venus	107,476,259	108,942,109	0.615197
Earth	147,098,290	152,098,232	1.000017
Mars	206,669,000	249,209,300	1.8808
Jupiter	740,573,600	816,520,800	11.8618
Saturn	1,353,572,956	1,513,325,783	29.4571
Uranus	2,748,938,461	3,004,419,704	84.323326
Neptune	4,452,940,833	4,553,946,490	164.79

Table 1: Orbital characteristics of the planets

where μ and p are parameters to planetary data derived from the Table 1.³ Note that these orbits are not circular but we can approximate them by circular orbits whose radius is the mean of the perihelion and aphelion. This is a very simplified model but gives surprisingly good results. Using 1,000,000 km as our unit of length and years as our unit of time, the known value of μ is 1.32197×10^8 . The value of μ determined from this data is 1.31368×10^8 . The value of p determined from this data is 1.4994.

As another exercise, students can compute the radius and altitude of a geostationary orbit – an orbit above the Earth’s equator whose period matches the (sidereal) period of the Earth’s rotation. A satellite in a geostationary orbit stays above the same point on the (rotating) Earth. Most commercial communications satellites use such orbits.

As mentioned above the best way to check your answer is by trying it in the simulation shown in Figure 1. The control panel for this simulation is shown in Figure 2. Notice the five meters at the top. The first three display the satellite’s current altitude (above the Earth’s surface), and current distance from the center of the Earth, current speed. During the simulation the next two display the minimum and maximum values of the distance from the center of the Earth. After the satellite has completed one full orbit these meters display the perigee and apogee of the orbit.

There are three digital controls and one push button control. The digital controls enable the user to enter the initial altitude and speed and the orbital inclination. The push button toggles the view from the bird’s eye view shown in Figure 1 to a view looking downward from the satellite toward the Earth. See Figure 3 shows one view from a satellite in a geostationary orbit. The view changes as night-and-day move across the Earth. You can also get different views depending on where you enter a geostationary orbit.

If you want to try it the altitude is 35,354.6 km and the speed is 3.09104 kps. If you enter these values in the appropriate boxes in the simulation and click the button to aim

³Source: Wikipedia.

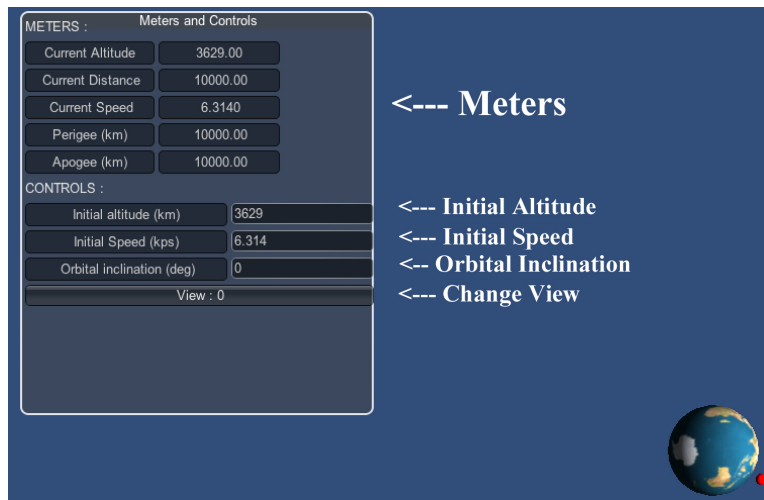


Figure 2: Simulation Control Panel

the camera downward from the satellite toward the Earth, you will see that the satellite remains stationary above the same point on the equator. Notice that as the Earth rotates about its axis we see night-and-day move across the earth.

5. Energy and Escape Velocity

This section is particularly appropriate when students study work and energy. Work and energy are, of course, two sides of the same coin – energy is the capacity to do work – and are measured in the same units – for example, kilogram meters² per second², or joules. In one dimension, the work exerted by a constant force, F , acting on an object that is displaced a distance, d , is Fd . If both force and displacement are vectors this formula becomes $\vec{F} \cdot \vec{d}$. In single variable calculus we study the work exerted by one dimensional forces that may not be constant

$$W = \int_a^b F dx$$

and in multivariable calculus this becomes

$$W = \int_P \vec{F} \cdot d\vec{r}.$$

There are two physical laws that are crucial to understanding orbits – conservation of

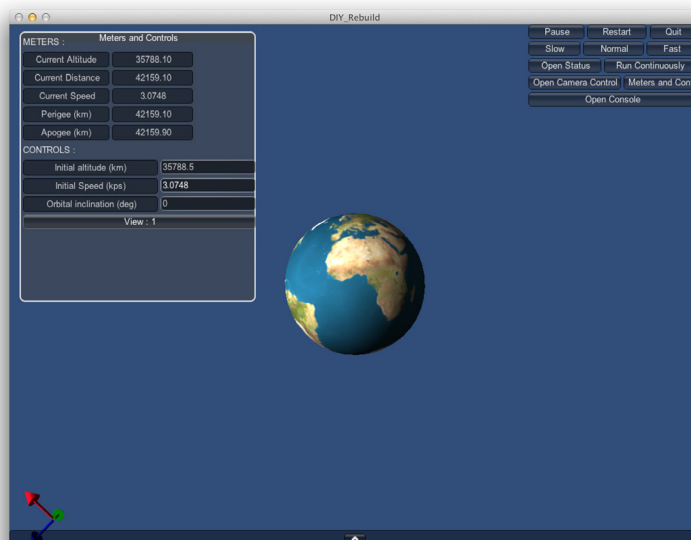


Figure 3: A View from a Satellite in a Geostationary Orbit

energy and conservation of angular momentum. In this section we look at conservation of energy. Two kinds of energy are of particular importance – kinetic energy and potential energy. The kinetic energy of an object of mass m traveling at a velocity v is

$$\text{kinetic energy} = \frac{mv^2}{2}$$

This is the usual form of the formula and it uses the word “velocity” where the word “speed” would be more precise. We will follow tradition and blur the distinction between speed and velocity – speed is a number and is the magnitude of velocity, which is a vector. This section uses only the one dimensional version and is appropriate for a single variable calculus class.

The formula for kinetic energy is well-known and most people don’t ask its origin. Deducing this formula from first principles is, however, an interesting exercise. Suppose that an object of mass m starts at rest and is subject to a constant force F from time zero to time T . Its acceleration is the constant

$$a = \frac{F}{m}.$$

So its velocity at time t is given by

$$v = \frac{Ft}{m}$$

and its velocity at time T is

$$V = \frac{FT}{m}$$

Its position at time t is

$$x(t) = \frac{Ft^2}{2m}$$

and

$$dx = \frac{Ft}{m} dt$$

Thus, the work done from time zero to time T is

$$\int_{x(0)}^{x(T)} F dx = \int_0^T \frac{F^2 t}{m} dt = \frac{F^2 T^2}{2m} = \frac{mV^2}{2},$$

the usual and well-known formula for kinetic energy.

Now suppose we want to determine the work done by gravity when an object moves straight upward from distance R_1 from the Earth to a distance R_2 .

$$W = \int_{R_1}^{R_2} F dr = \int_{R_1}^{R_2} -\frac{\mu m}{r^2} dr = \frac{\mu m}{R_2} - \frac{\mu m}{R_1}.$$

This is an easy exercise in integration. The difference in the potential energy of an object of mass m moved from a distance R_1 to a distance R_2 is

$$\Delta P = \frac{\mu m}{R_1} - \frac{\mu m}{R_2}.$$

Notice that when the object is higher – that is, $R_2 > R_1$ – it has more potential energy because when it falls it will gain kinetic energy due to the force exerted by the Earth's gravity.

If we want to escape completely from the Earth then R_2 is infinite and to determine the work done by gravity we must evaluate the improper integral

$$W = \int_{R_1}^{+\infty} F \, dr = -\frac{\mu m}{R_1}.$$

Because the work done by gravity is negative, by conservation of energy, the energy to escape from the Earth must come from somewhere. If, for example, we want to launch a rocket from the surface of the Earth and have it escape from the Earth the required energy is

$$\frac{\mu m}{R_1} = \frac{3.98669 \times 10^{14} m}{6.371 \times 10^6} = 62,575,500m.$$

If the rocket is launched with an initial velocity v then its kinetic energy will be

$$\frac{mv^2}{2}$$

and it will just barely escape from the Earth if

$$\begin{aligned} \frac{mv^2}{2} &= 62,575,500m \\ \frac{v^2}{2} &= 62,575,500 \\ v &= \sqrt{125,151,000} \\ &= 11,187 \text{ meters per second} \\ &= 11.187 \text{ kps} \end{aligned}$$

This is often called the *escape velocity*. As always, this is a simplification. Students should discuss generally the impact on this result of air resistance and of the initial velocity that comes from the Earth's rotation. This is a good place to mention that the Kennedy Space

Center is located relatively near the equator to take advantage of the initial velocity that comes from the Earth's rotation.

6. Black Holes and a Not So Picky Point

In the last section we essentially computed the difference in potential energy for an object of mass m between a distance R_1 and a distance R_2 from the center of the Earth

$$\Delta P = \int_{R_1}^{R_2} F \, dr = \int_{R_1}^{R_2} \frac{\mu m}{r^2} \, dr = \frac{\mu m}{R_1} - \frac{\mu m}{R_2}.$$

by computing the work required to move it from R_1 to R_2 . Looking at the formula one might be tempted to say the object's potential energy at a distance R is given by

$$P = -\frac{\mu m}{R}$$

but this formula doesn't make sense. One problem is that this formula predicts the energy is infinite if $R = 0$. The deeper problem is that we must measure energy relative to something. The difference in energy of an object between two different heights does make sense, but the energy of an object at a particular height does not. The fact that the formula predicts infinite energy if $R = 0$ is essentially the reason that black holes exist.

7. The "Energy" of an Object in a Circular Orbit

The quantity

$$P = -\frac{\mu m}{R}$$

that can be used to compare the potential energy of an object of mass m in different orbits is only part of the story. We also need to consider kinetic energy. The kinetic energy of an object of mass m in a circular orbit of radius R is

$$\frac{ms^2}{2} = \frac{\mu m}{2R}$$

using our earlier determination of the orbital speed. Thus, if an object moves from a circular orbit of radius R_1 to a circular orbit of radius R_2 the change in its kinetic energy is

$$\Delta K = \frac{\mu m}{2R_2} - \frac{\mu m}{2R_1}$$

Putting this together with our earlier computation of the difference in potential energy we get the following formula for the difference in total energy

$$\Delta E = \Delta K + \Delta P = \frac{\mu m}{2R_2} - \frac{\mu m}{2R_1} + \frac{\mu m}{R_1} - \frac{\mu m}{R_2} = \frac{\mu m}{2R_1} - \frac{\mu m}{2R_2}$$

Notice that for a higher orbit the total energy is larger.

Even though the quantity

$$E_R = -\frac{\mu m}{2R}$$

can only be used when we are comparing two different circular orbits, we often loosely call this quantity the *fictitious total energy* for a circular orbit of radius R . The negative sign is needed because the energy required to move from a circular orbit of radius R_1 to a circular orbit of radius R_2 is

$$E_{R_2} - E_{R_1} = \frac{\mu m}{2R_1} - \frac{\mu m}{2R_2}.$$

Notice that energy is required to move from a lower circular orbit to a higher circular orbit, since $R_2 > R_1$.

8. Orbits and Trajectories

So far we have discussed two kinds of trajectories – circular orbits and trajectories straight upward from the surface of the Earth. Now we look at more general kinds of trajectories. These are typically studied in a section on Kepler's Laws and build on the material covered on conic sections and on vector operations like the cross product. Trajectories for a satellite around a central body like the Sun or the Earth must be conic sections – hyperbolas, ellipses, or parabolas. The orbits we are interested in are ellipses. One standard form for an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a \geq b > 0$. See Figure 4. Notice if $a = b$ then this is just a circle of radius a .

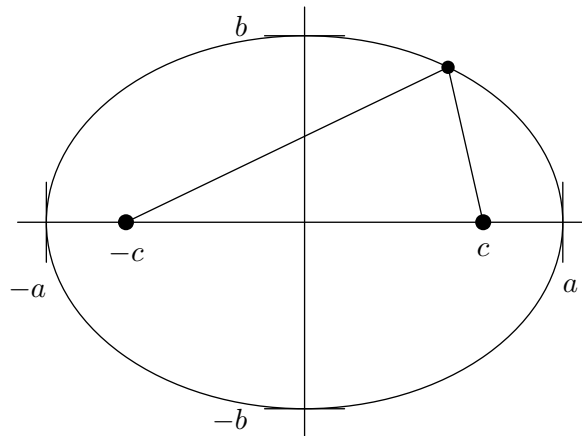


Figure 4: An Ellipse in Standard Form

This ellipse has two *foci* – at the points $(-c, 0)$ and $(c, 0)$ where

$$c = \sqrt{a^2 - b^2}.$$

and two interesting geometric properties.

- A light ray emanating from one focus and bouncing off the ellipse will be reflected back to the other focus.
- The total distance from one focus to the ellipse and back to the other focus is constant.

For a satellite in an elliptical orbit around a central body the central body is at one focus. Thus, the apogee and perigee⁴ are

$$\text{perigee} = a - c \quad \text{and} \quad \text{apogee} = a + c.$$

Notice that

$$a = \frac{\text{perigee} + \text{apogee}}{2}.$$

⁴Aphelion and perihelion if the Sun is the central body.

For an elliptical orbit the total fictitious energy is

$$E = -\frac{\mu m}{2a} = -\frac{\mu m}{\text{perigee} + \text{apogee}}$$

Notice that if the orbit is circular then a is just the radius and this becomes our earlier formula.

Earlier in this paper we developed the formula

$$\left(\frac{2\pi}{\sqrt{\mu}}\right) r^{3/2}$$

for the period of a circular orbit. For an elliptical orbit the formula is

$$\left(\frac{2\pi}{\sqrt{\mu}}\right) a^{3/2}$$

Notice that if the orbit is circular then a is just the radius and this becomes our earlier formula.

We often want to move a satellite from one circular orbit to another circular orbit – for example, a satellite is often put into a low “parking orbit” and later raised to a higher orbit. Figure 5 shows a maneuver, called “Hohmann Transfer,” that is commonly used. This maneuver requires two burns:

- A short first burn that applies a force in the direction the satellite is traveling. Thus, it increases the speed of the satellite but not the direction in which it is traveling. This puts the satellite in an elliptical orbit whose perigee is the radius of the original orbit and whose apogee is the radius of the higher orbit. This orbit is called the “Hohmann transfer orbit.”
- A second short burn when the satellite reaches apogee. This burn changes the kinetic energy to “circularize” the orbit.

Suppose we have a satellite of mass m in a circular orbit 10,000 km from the center of the earth, or at an altitude of 3,629 kilometers above the Earth’s surface and we want to change its orbit to a geostationary orbit. Solving this problem requires many steps but we’ve covered all the key ideas. Students can and should check their final answer in a simulation. Because this solving this problem requires many steps with many opportunities

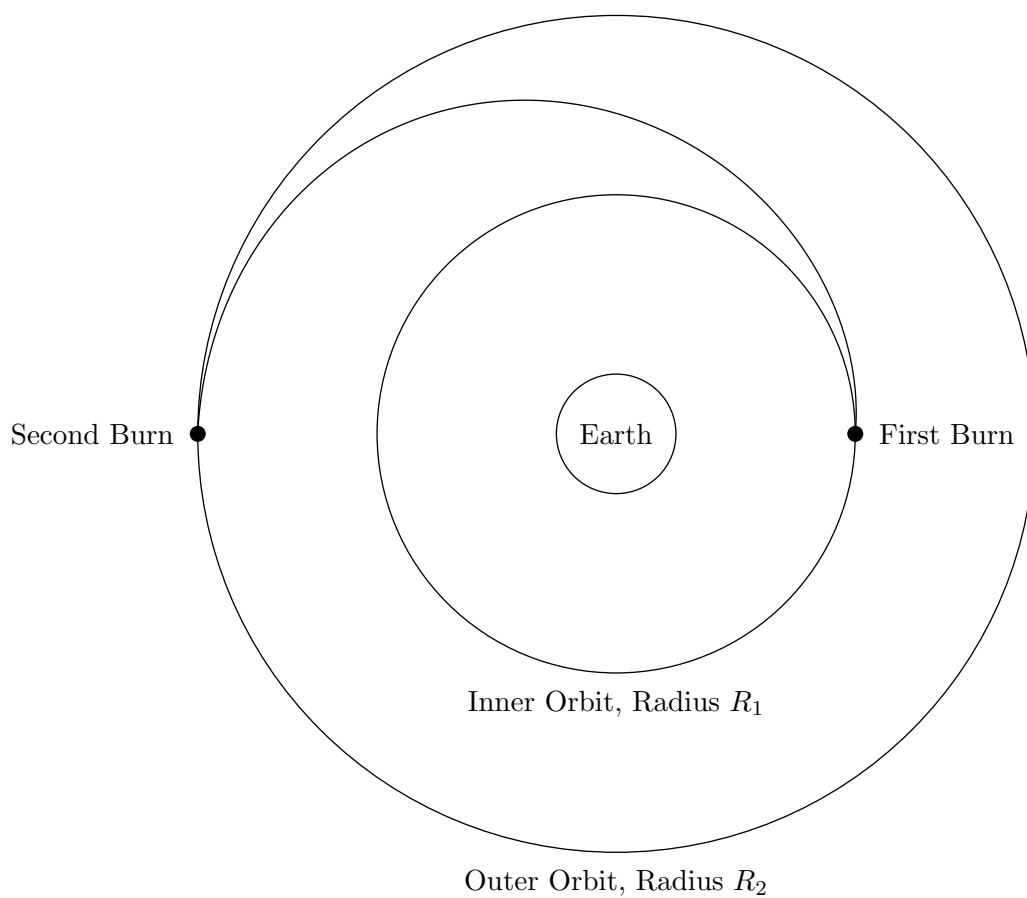


Figure 5: Hohmann transfer – changing orbits

for errors, checking the work in a simulation is particularly powerful. You can set this up for students at various levels by giving them an appropriate series of questions that lead them to a solution. The questions below provide one example.

Question 1 *What is the total energy required to change orbits?*

Question 2 *What is the satellite's speed in its current orbit?*

Question 3 *What is the satellite's speed in the geostationary orbit?*

Question 4 *The Hohmann transfer orbit is an elliptical orbit whose perigee is the radius of the original orbit and whose apogee is the radius of the geostationary orbit. Find a and b for the Hohmann transfer orbit. What is the additional energy required to change the orbit from the original orbit to the Hohmann transfer orbit?*

Question 5 *What is the required change in velocity for the first burn – to achieve the Hohmann transfer orbit?*

Question 6 *What is the satellite's velocity immediately after the first burn?*

Question 7 *What is the satellite's velocity when it reaches the apogee of the elliptical orbit?*

Question 8 *When will the satellite reach the apogee of the elliptical orbit?*

Question 9 *What is the required change in velocity for the second burn – to circularize the orbit? Is it in the direction the satellite is traveling or exactly opposite to that direction?*

You can also ask many additional questions that build on the ideas we've developed in this paper – for example,

Question 10 *Suppose you want to place the satellite in a geostationary orbit above a particular point on the Earth's equator. Choose your point and time the burns to achieve the desired orbit.*

Question 11 *Suppose we want to visit Mars. Find the Hohmann transfer trajectory from Earth to Mars. How long will it take to go from Earth orbit to Mars orbit?*

Question 12 *Find the Hohmann transfer trajectory to return from Mars orbit to Earth orbit.*

Question 13 *Compare the energy required for the outbound trip to the energy required for the return trip.*

Question 14 *Discuss when a rocket should leave the vicinity of the Earth in Earth orbit to arrive in Mars orbit near Mars.*

Question 15 *Discuss when a rocket should leave the vicinity of Mars in Mars orbit to arrive in Earth orbit near Earth.*

Question 16 *How long would a round trip from Earth to Mars and back require?*