HELICOPTERS AND DRONES – ENGAGING MATHEMATICS

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Figure 1 shows the Parrot quad rotor drone sold for under $300.00 by Brookstone, Barnes and Noble, Amazon.com, and others. Students, faculty, children, grandchildren, nieces, nephews, and passersby love playing with this drone. We are interested in it here because it is a great setting in which to learn mathematics and science. We have been using it at the United States Military Academy (West Point) with great success and excitement in math classes and in a summer program for high school students interested in coming to West Point.[1]

[1] Please see http://diymodeling.appstate.edu/node/66 for associated simulations.

Figure 1: Flying the Parrot quad rotor drone with the indoors hull

The Parrot drone is designed for both indoor and outdoor use. Figure 1 shows the drone with its indoor hull and Figure 2 shows it with its outdoor hull. The indoor hull protects the rotors and makes it much safer to fly both indoors and outdoors. The drone's four rotor design and controls make it very easy to fly using either an iOS (iPad or iPhone) or Android device. Our drone has survived well over 300 high school students, 100 college students, 50 faculty, innumerable children, grandchildren, nieces, nephews, friends, passersby, and two big trees.

The drone has two cameras that can record both still photographs and movies and also stream live video during flight to the controlling iOS or Android mobile device. Figure 3 shows one frame from a video taken by the forward-facing camera. The second camera faces downward.

The Parrot Drone can be used to engage students in real problems that involve standard topics covered in the usual science and mathematics courses at the high school and college level. Here is a very partial list of the science and mathematics topics that make the Parrot drone work.

Figure 2: Flying the drone with the outdoor hull

Figure 3: An aerial photograph taken from the forward-facing camera

- Mathematics
  - Addition, Subtraction, Multiplication, Division, and Circles.
  - The Pythagorean Theorem and Trigonometry.
  - Vectors.
  - Linear Algebra and Coordinate Transformations.
  - Differential Equations.
  - Fluid Flow and Airfoils.
  - Statistics.
- Physics
  - Speed, Velocity, Acceleration, and Force.
  - Work and Energy
  - Torque and Angular Momentum.
  - Air Pressure.
- Chemistry – Batteries.

In addition to flying drones and full-sized helicopters, many of these ideas apply to other problems – for example, the design of wind turbines. We look at some examples of how the Parrot drone can be used in math and science classes below.

1. First Look

Students at almost any level can begin with two basic questions:

Question 1 *How fast can the Parrot drone fly?*

Question 2 *How long can the Parrot drone fly on a single battery charge?*

The natural way to answer these questions is by experimentation. Students can work in teams of two, taking turns flying a pre-determined course and recording elapsed time for each flight and total flight time on a single charge. These experiments will naturally stimulate many additional questions. These experiments are best done inside in a large room but they can be done outside on a calm day.

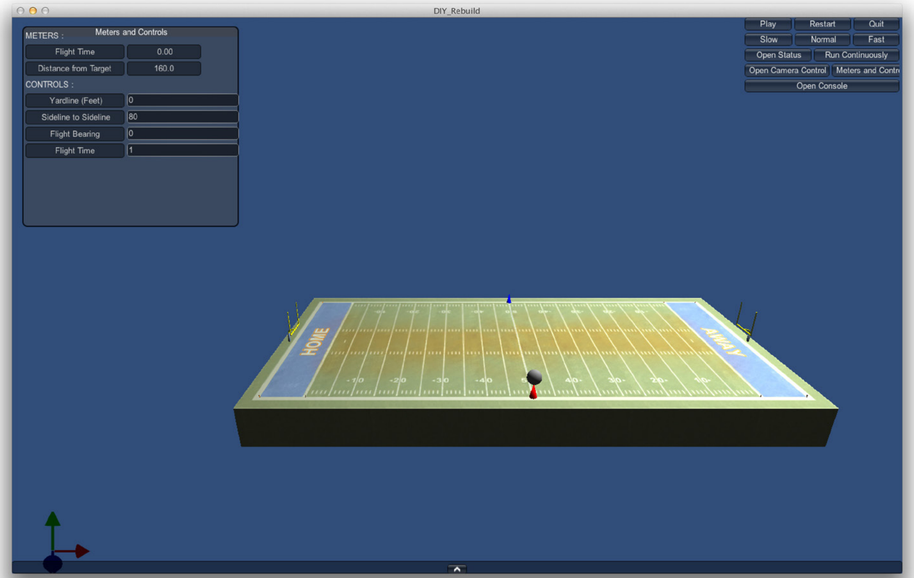

Figure 4: A simulation for checking distance and bearing problems

2. Flight-Planning and Trigonometry

In this set of problems, instead of checking their work by looking in the back of the book or getting it marked right or wrong by the teacher or a computer-based grading system, students check their work by seeing if they get the right results in a simulation like the one shown in Figure 4. We have found that students are much more likely to persevere and correct mistakes when they check their work in this way. This simulation was built using software created by the **DIYModeling** (Do It Yourself Modeling) project funded by the National Science Foundation.[2] To learn more about this project and for the free **DIYModeling** software and simulations please visit http://diymodeling.appstate.edu/node/66.

These simulations take place on a football field. Although we usually talk about the dimensions of a football field in yards, the width of the field is 160 feet. For that reason we use feet instead of yards. In Figure 4 and Figure 5 the drone starts at a base on the 50 yard line on the near sideline. The 50 yard line runs from south to north with the far side line being north. The drone flies at five feet per second. The missions involve flying under various wind conditions from the base to a destination. In Figures 4 and 5 the destination is on the far sideline on the 50 yard line.

[2] Award number 0919264.

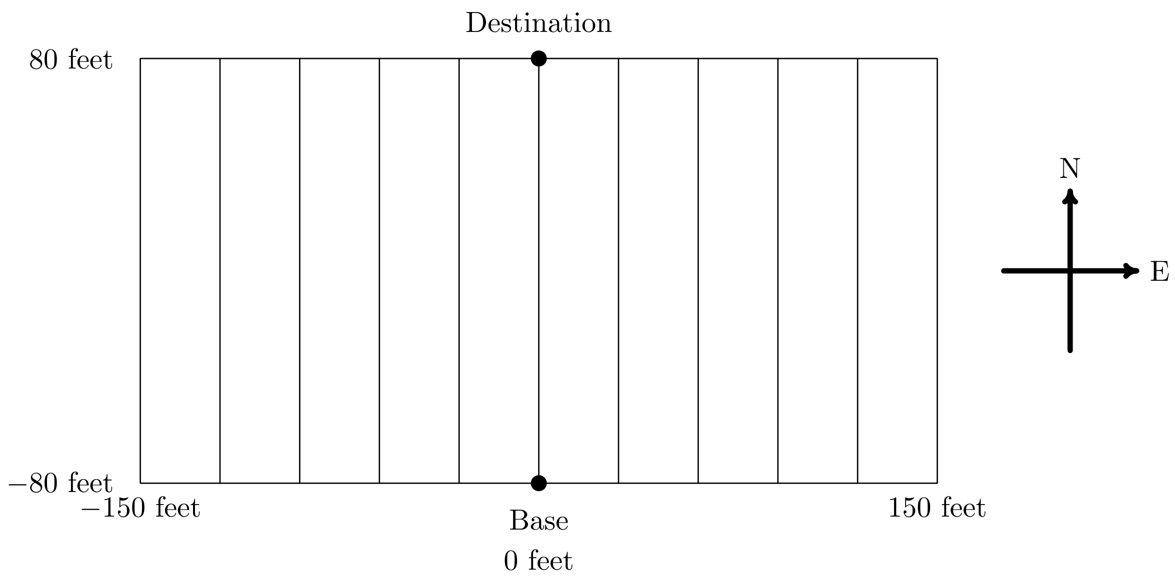

Figure 5: Flight Planning, I

Students describe each flight by completing a table like Table 1.

1. Base x	0
2. Base y	−80
3. Destination	0
4. Destination y	80
5. Flight Bearing	0
6. Flight Distance	160
7. Flight Time	32

Table 1: Flight-Planning Table

The first two entries are the x- and y-coordinates (in feet) of the base. The next two entries are the $x-$ and $y-$coordinates (in feet) of the destination. The fifth entry is the bearing, measured in degrees clockwise from North, from the base to destination. See Figure 6. The sixth entry is the distance, measured in feet, from the base to the destination. The seventh entry is the flight time, or duration, measured in seconds. Figure 7 shows the area in the simulation where students enter the four key entries in the table above. The x- and y-coordinates of the destination are entered in the boxes labeled `Yardline (feet)` and `Sideline to Sideline` respectively. The flight bearing and flight time entries are made in the boxes labeled `Flight Bearing` and `Flight Time`.

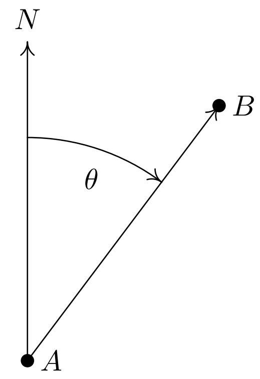

Figure 6: Bearing

Figure 7: Distance and bearing simulation control panel

After entering all this information the students clicks the buttons – `Restart` and `Play`, watches the simulation, and checks that the drone reaches the intended destination at the time the flight is completed. A meter tracks the distance from the drone to the destination, or target.

Question 3 *Plan and execute (in the simulation) a flight from the base at* $(0, -80)$ *to a destination on the far sideline and the fifty yard line:* $(0, 80)$.

Question 4 *Plan and execute a flight from the base to the home team goal line on the far side line.*

Question 5 *Plan and execute a flight from the base to the home team goal line midway between the two sidelines.*

Question 6 *Plan and execute a flight from the base to the away team's 30 yard line on the far side line.*

3. Addition, Division, the Pythagorean Theorem, and Vectors

For the next set of questions students look at a more complicated mission – an out and return flight from the base to a destination under various wind conditions. We begin with some warm-up questions.

Question 7 *Suppose you want to fly from a base to a destination that is 160 feet due north of the base and return. As usual the drone flies at five feet per second.*

- *How long will the round-trip take if there is no wind?*
- *What would be the impact on the total flight time of a two foot per second wind blowing from north to south?*

This question usually provokes lively debate with many students arguing that the wind has no effect on the total flight time because the time lost on the outbound leg because of the headwind will be recovered on the return leg because of the tail wind.

The next question clarifies the answer to the preceding question and can be checked in a simulation. See Figure 8 and Figure 5.

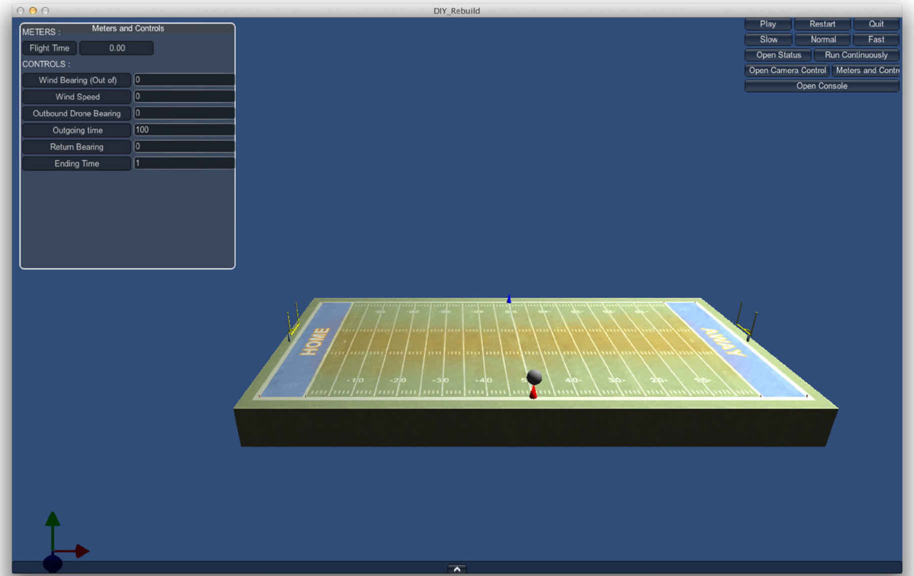

Figure 8: Flight Planning Simulation

Question 8 *How long would the outbound leg take? How long would the return leg take? What would happen if the wind speed was six feet per second?*

First Leg Bearing	
Elapsed Time at Destination	
Second Leg Bearing	
Total Elapsed Time at Base	

Note that the preceding questions require only simple arithmetic. The next question can be answered using the Pythagorean theorem, trial-and-error, or vectors.

Question 9 *Plan this mission with a 3 feet per second wind blowing from east to west. Check your work using the simulation.*

The next question requires trail-and-error or vectors.

Question 10 *Plan this mission with a 3 feet per second wind blowing from northeast to southwest. Check your work using the simulation.*

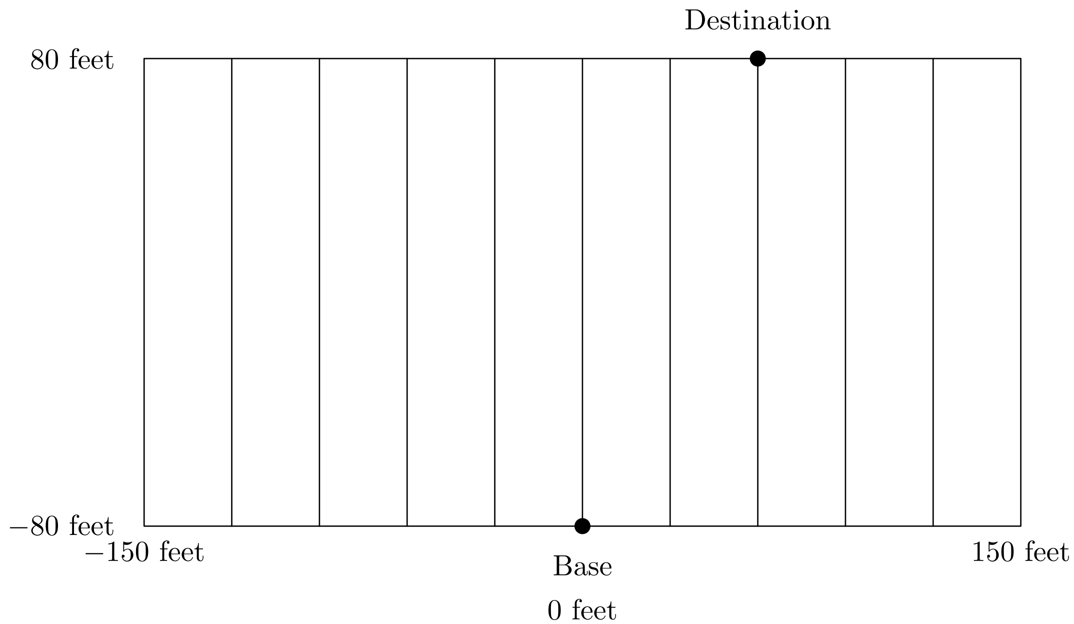

Figure 9: Flight Planning, II

You can create additional problems using the base and destination shown in Figure 9 and Figure 10.

4. Position & Velocity in 2-Dimensions, the Language of Differential Equations

After doing problems like those above, it is natural to ask what would happen if a pilot who didn't know any mathematics flew in the most naive way – simply pointing the plane directly toward a destination. Figure 11 shows several still frames from a simulation in which such a pilot is flying from the point $(-150, 0)$ on one goal line to the point $(150, 0)$ on the other goal line with a three foot per second wind blowing from the south to the north. This flight takes 93.8 seconds. A mathematically inclined pilot would do the same flight in 60 seconds.

Students can simulate a non-mathematical pilot using **DIYModeling**[3] software. We use two simulations.

- A "flight simulator" in which the player controls the direction the drone flies.
- A skeleton simulation that the student must complete.

[3] As usual see our web site http://diymodeling.appstate.edu/node/66

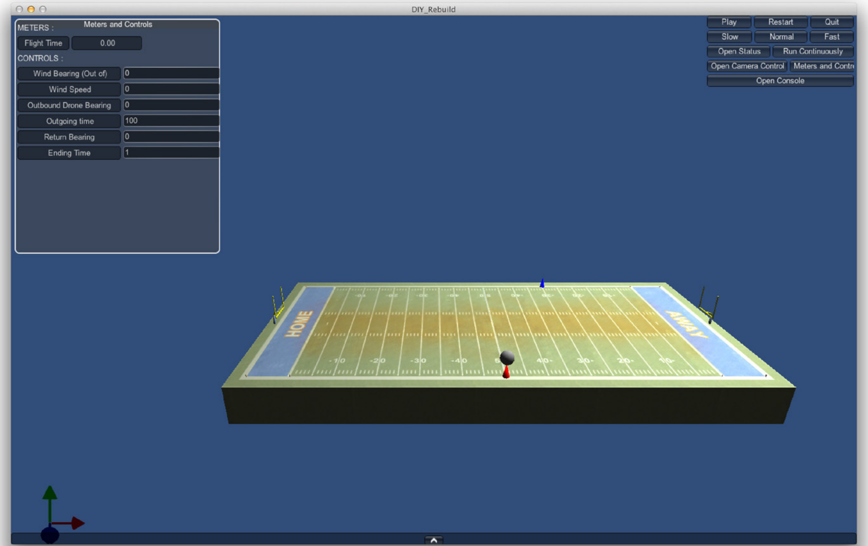

Figure 10: Another Flight Planning Simulation

For the second simulation, students must describe the flight of a non-mathematical pilot using the language of differential equations. We let $\vec{p}$ denote the position of the drone or plane and $\vec{d}$ denote the position of the destination. Students must supply the differential equation:

$$\vec{p}' = \left(\frac{5}{||\vec{d} - \vec{p}||} \right) (\vec{d} - \vec{p}).$$

This is a great exercise in the language of differential equations for vector-valued functions.

5. Frames-of-Reference

The Parrot drone can be flown in two different ways, or modes. If you are able to watch the drone in flight, then the easiest mode is called "line-of-sight" or "absolute" mode. If you are unable to watch the drone because of distance or obstructions, then you can fly it in "relative" mode using the view streamed from the forward facing camera. See Figure 3.

It is important to think of two different frames-of-reference. One frame-of-reference is defined by the device, iOS or Android, controlling the drone. These devices know where they are and using an electronic compass and three axis accelerometer they know how they

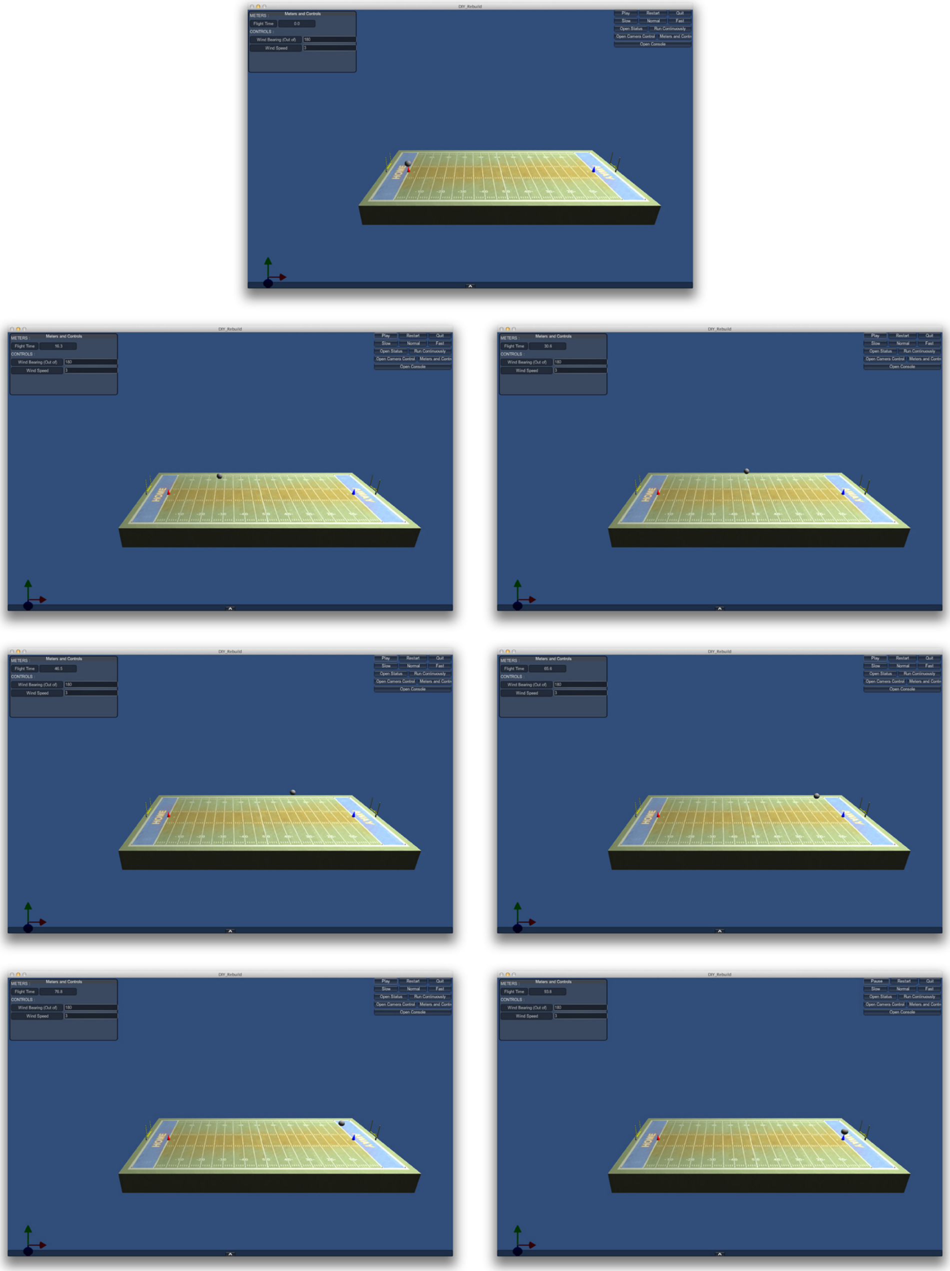

Figure 11: Stills from a simulation

are being held. The second frame-of-reference is attached to the drone with the forward direction defined by the direction in which the forward-facing camera is pointing.

In absolute mode, when you tilt the controlling device the drone tilts the same way in the same frame-of-reference as the controlling device. This causes the drone to fly in the same direction as you tilted the controlling device. In relative mode when you tilt the controlling device the drone tilts the same way in the frame-of-reference attached to the drone. When you tilt the controlling device downward away from you then the drone flies in the direction in which the forward fixing camera is pointing. In effect, you can think of you and your controlling device as sitting in the pilot's seat – or more accurately the seat in which the pilot would be sitting if the drone were manned rather than unmanned.

In the first semester of the four semester core mathematics sequence taken by almost all our cadets, we look at frames of reference in the context of interpreting aerial photographs and translating back-and-forth between coordinates on a photograph and ground coordinates. The drone, with its photographic capabilities, brings up these questions in a natural way.

If an aerial photograph is taken by the downward facing camera while the drone is flying level, then the transformation from coordinates, x_1 and x_2, on the photograph to coordinates, y_1 and y_2, on the ground (or a map of the ground) are given by a transformation of the form

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + b_1 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + b_2 \end{aligned}$$

or, in vector-and-matrix notation,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \text{or} \quad \vec{y} = A\vec{x} + \vec{b}.$$

This brings up many interesting problems for classes covering matrices and vectors – for example,

Question 11 *Given the transformation above from photograph coordinates to ground coordinates, find the transformation from ground coordinates to photograph coordinates.*

Question 12 *Given the ground and photograph coordinates of three recognizable features, find the transformation that converts photograph coordinates to ground coordinates.*

6. Data-Gathering and Analysis

The limiting factor for flights of the Parrot drone is battery capacity. The drone uses fairly heavy batteries with a flight time of about ten minutes. The batteries run hot and their performance is easily degraded by hot operation. We suggest numbering your batteries if you have more than one and keeping records of their performance on each flight. Battery performance is a hot topic – for example, the success of all-electric vehicles will depend on batteries. As these words are being written, lithium polymer batteries are in the news because of battery problems in Boeing's Dreamliner.

There are many natural questions that might be answered by careful record keeping and analysis.

Question 13 *Is there a difference in battery performance between the outdoor hull and the indoor hull?*

Question 14 *How does battery performance change as the batteries are used?*

With our first batteries we have noticed a dramatic drop in flight time before the controller indicates low battery. The battery life seems to drop less than the indication. As one result, with older batteries we have to avoid landing because the controller may prevent take-off because of apparent low charge.

Question 15 *How do flight characteristics impact flight time – for example, if the drone ascends and descends often does that shorten flight time and by how much?*

Question 16 *What is the drone's top speed? How is the top speed affected by the choice of hull?*

Question 17 *How does the speed of the drone affect flight time and distance traveled?*

Question 18 *How rapidly can the drone reverse direction?*

Question 19 *How heavy a load can the drone lift? What is the impact of the load on speed and flight time?*

One way to collect data for the last two questions is by making and analyzing video-recordings.

7. Addition, Subtraction, and the Perimeter of a Circle

This set of questions begins with a simple experiment – students hold a 3 by 5 card in front of a fan with at least two different speeds and discuss the effects of wind speed and the angle at which the card is held relative to the wind. This angle is called the "angle of attack." Then they discuss the impact speed has on the best angle of attack for an airplane wing. They should come up with several observations.

1. For an airplane wing, lift – the force pushing upwards – is good. It keeps the plane in the air and the more lift the higher the weight that the airplane can carry.
2. For an airplane wind, drag – the force pushing backwards – is bad. It makes the engines work harder and drains the battery.
3. Varying the angle of attack of the wing (3 by 5 card) changes the lift and drag. At high speeds a shallow angle of attack is better because it minimizes drag while producing sufficient lift. At lower speeds a steeper angle of attack is necessary to provide sufficient lift. Students who have flown in airplanes may have observed how the wings' shapes are changed by the flaps for landing and take-off.

Question 20 *Suppose a helicopter's rotor(s) are turning at a fixed number of revolutions per minute – compare the air speed of the tips of the rotor blades with the air speed of the parts of the rotor blades closer to the center of the rotor. How might you design rotor blades based on this comparison? Look at the blades on the drone's rotors. Describe what you see.*

Question 21 *Suppose that a rotor is revolving clockwise as viewed from above and that the speed of the tips of the rotor blades is 30 feet per second when the helicopter is hovering. Now suppose the air speed of the drone is 5 feet per second. How does this affect the air speed of the tips of the rotor blades? The blades that are revolving in the same direction as the drone are called "advancing" blades. The blades that are revolving in the opposite direction are called "retreating" blades. Should the advancing and retreating blades have the same angle of attack?*

Question 22 *Suppose that the advancing and retreating blades for a single rotor helicopter have the same angle of attack. How will this affect the helicopter's flight.*

Question 23 *The quad rotor Parrot drone has four rotors. Based on your answer to the question above do you think they should all revolve in the same direction? Do they?*

Full-sized helicopters vary the pitch of the blades as they revolve.

8. Traveling Salesman Problems on Steroids

The classical traveling salesman problem asks for the shortest route a traveling salesman can follow to visit a set of customers. For the drone, we often have a set of targets – sites where we'd like to take aerial photographs – and would like to visit as many of them as possible on a single battery charge. The targets may have different values or priorities and the problem may the further complicated by wind. Even the simplest traveling salesman problems are very difficult and our more complicated problems are even more difficult. These are great problems to give students.

One possible way of using these problems in class would be to have a two-stage contest with teams. Every team would participate in the first stage and check their solution using a simulation. The highest scorers from the first stage would move on to the second stage using real drones. They would have to adjust their plans in real time depending on wind conditions. The second stage would require flying ability as well as planning ability.

9. Torque and Angular Momentum

These are standard topics in both mathematics and physics and often cause students great difficulty. Rotary wind aircraft are inundated by torque and angular momentum and this is a great venue in which to study these topics. One reason that four rotor helicopters are so stable is that with two rotors revolving counterclockwise and the other two revolving clockwise, a lot of the torque is cancelled out.

10. Creative Missions

Figures 12 and 13 show a map and aerial photograph of an "area-of-operations" at West Point. We gave these materials to students and gave them an open-ended assignment to plan a mission using the drone. Interestingly, many of the missions involved getting food. Many of the students were very creative. A project of this type will be more meaningful if you choose an interesting area of operations. Google Earth is a great source of aerial images.

Figure 12: An aerial photograph of the area of operations


Legend
Road Centerline
Buildings
Parking Areas
Roads
Installation Boundary
Lakes and Ponds
Hudson River
HUDSON RIVER
Constitution Island
Target Hill Athletic Fields
Shea Stadium
North Athletic Field
North Dock
Trophy Point
The Plain
River Courts
Delafield Pond
Lusk Reservoir
Michie Stadium
South Dock
Adolphs Pond
Stony Lonesome Family Housing
Stony Lonesome Gate
Dassori Pond
Buffalo Soldiers Field
Thayer Gate
WEST POINT
WEST POINT, NY
NORTH
SCALE: 1" = 500'
500 250 0 500 1,000 1,500 2,000 Feet