USING THE TI-89 TO DELVE DEEPER INTO THE GRAPHS OF POLYNOMIAL FUNCTIONS

Jay L. Schiffman

Rowan University

201 Mullica Hill Road

Glassboro, NJ 08028-1701

schiffman@rowan.edu

Abstract: With the aid of the TI-89, we consider the relationship between the general cubic and location of extrema and inflection points, the connection between a cubic having distinct real roots and the abscissa of the inflection point, the link between multiple roots and the derivative, and the abscissa of the vertex of a quadratic having distinct real roots.

For our first application, we consider the general cubic polynomial $p(x) = a \cdot x^3 + b \cdot x^2 + c \cdot x + d;\ a, b, c, d \in \mathbb{R}$ *and* $a \neq 0$. We secure the relative extrema and point of inflection via the TI-89 / VOYAGE 200 in **FIGURES 1-20:**

FIGURE 1: The define command.

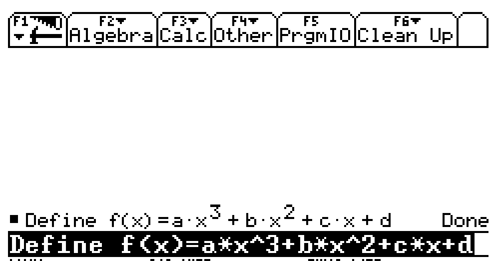


FIGURE 2: The general cubic defined.

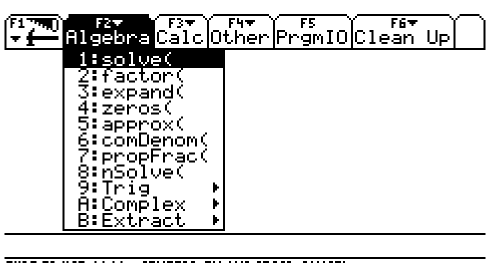


FIGURE 3: The solve command.

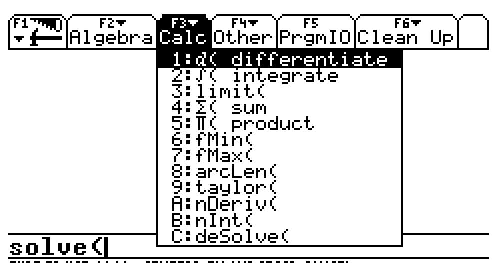
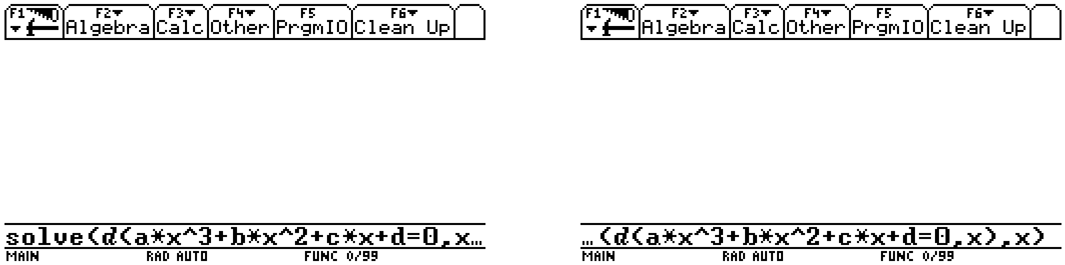
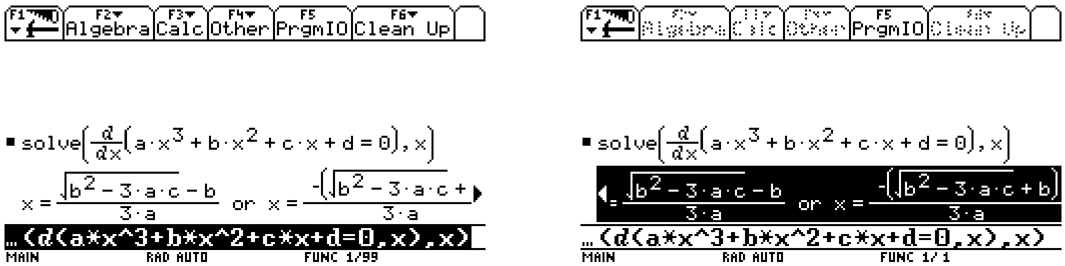


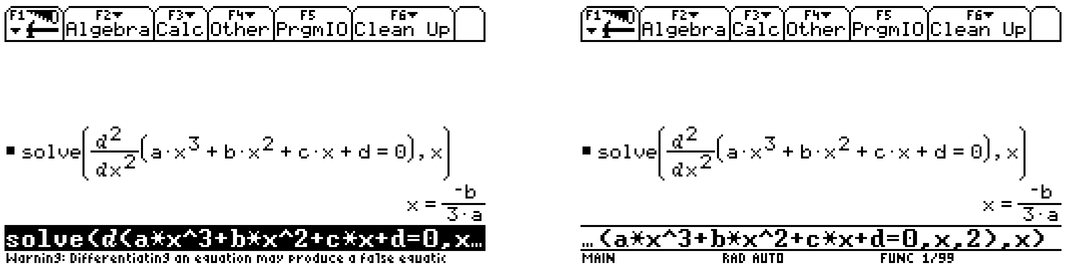
FIGURE 4: The derivative command.



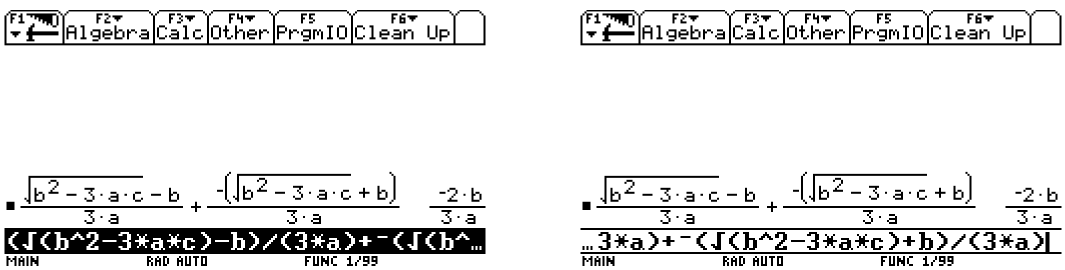
FIGURES 5 and 6: The function input.



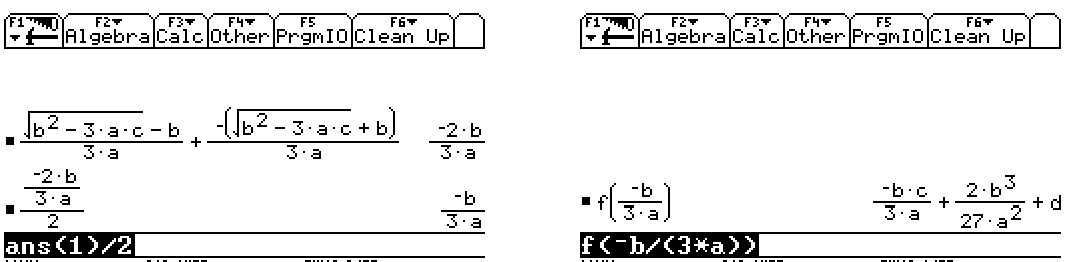
FIGURES 7 and 8: The critical numbers are secured.



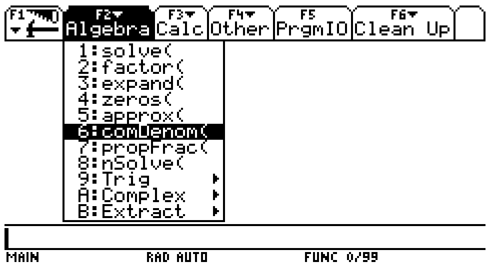
FIGURES 9 and 10: Securing the abscissa of the point of inflection.



FIGURES 11 and 12: The sum of the critical numbers is found.



FIGURES 13 and 14: The average of the critical numbers coincides with the abscissa of the point of inflection.



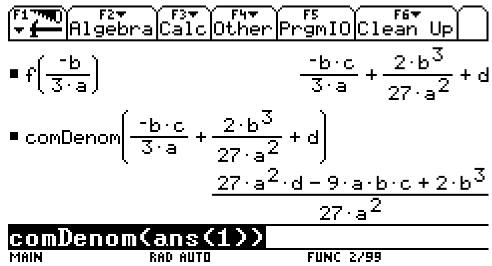
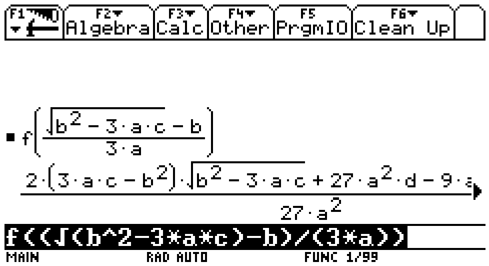
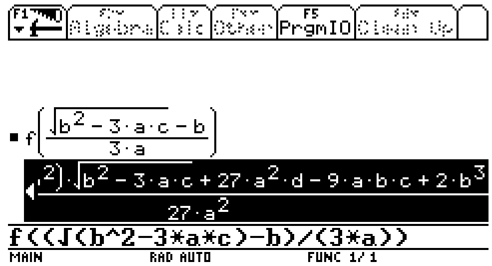
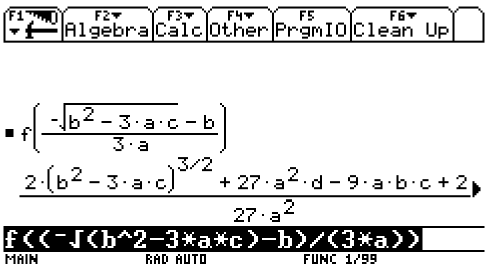


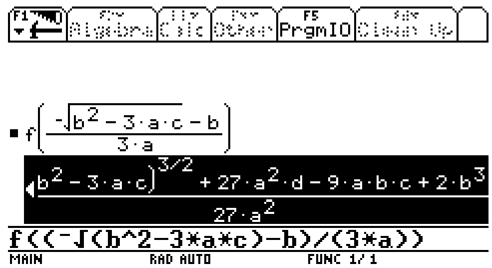
FIGURE 15: The common denominator command. FIGURE 16: Use of the common denominator command to combine the ordinate of the inflection point into a single fraction.





FIGURES 17 and 18: Securing the ordinate of the first critical number.





FIGURES 19 and 20: Securing the ordinate of the second critical number.

If one takes the average of the two ordinates of the relative extrema, they obtain the ordinate of the point of inflection as seen in **FIGURE 21:**

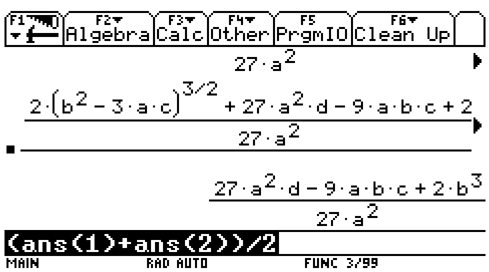


FIGURE 21: The average of the ordinates of the critical numbers coincides with the ordinate of the inflection point.

In our second application, we consider a cubic polynomial having three distinct real roots and show that the abscissa of the point of inflection is precisely the average of these three real zeros. To prove this, let the three distinct x intercepts be r_1, r_2 *and* r_3 respectively. Then by the factor theorem, $p(x)=a\cdot(x-r_1)\cdot(x-r_2)\cdot(x-r_3);\ a\neq 0.$ Then using the formula for the derivative of a product of three differentiable functions, one obtains

$$p'(x)=\left[a\cdot(x-r_1)\cdot(x-r_2)\cdot(x-r_3)\right]'=a\cdot\left[(x-r_1)\cdot(x-r_2)\cdot(x-r_3)\right]'=$$
$$a\cdot\left[(x-r_1)\cdot(x-r_2)\cdot(x-r_3)'+(x-r_1)\cdot(x-r_2)'\cdot(x-r_3)+(x-r_1)'\cdot(x-r_2)\cdot(x-r_3)\right]=$$
$$a\cdot\left[(x-r_1)\cdot(x-r_2)+(x-r_1)\cdot(x-r_3)+(x-r_2)\cdot(x-r_3)\right].$$
$$p''(x)=a\cdot\left[(x-r_1)\cdot(x-r_2)+(x-r_1)\cdot(x-r_3)+(x-r_2)\cdot(x-r_3)\right]'=$$
$$a\cdot\begin{bmatrix}(x-r_1)\cdot(x-r_2)'+(x-r_2)\cdot(x-r_1)'+(x-r_1)\cdot(x-r_3)'+(x-r_3)\cdot(x-r_1)'+(x-r_2)\cdot(x-r_3)'+\\(x-r_3)\cdot(x-r_2)'\end{bmatrix}$$
$$=a\cdot\left[(x-r_1)+(x-r_2)+(x-r_1)+(x-r_3)+(x-r_2)+(x-r_3)\right]=a\cdot\left[6\cdot x-2\cdot r_1-2\cdot r_2-2\cdot r_3\right]$$
$$p''(x)=0\Leftrightarrow a\cdot\left[6\cdot x-2\cdot r_1-2\cdot r_2-2\cdot r_3\right]=0\Leftrightarrow 6\cdot x-2\cdot r_1-2\cdot r_2-2\cdot r_3=0\Leftrightarrow 6\cdot x=2\cdot r_1+2\cdot r_2+2\cdot r_3\Leftrightarrow$$
$$6\cdot x=2\cdot(r_1+r_2+r_3)\Leftrightarrow x=\frac{r_1+r_2+r_3}{3}.$$

For our third application, consider a general polynomial p having a multiple root @ $x=r.$ Let m be the multiplicity of this root. We show that in this situation, $p\ and\ p'$ share a non trivial common factor and conversely. We argue as follows:

We first assume that p has a multiple root r of multiplicity $m.$ Then $p(x)=(x-r)^m\cdot q(x)$ using the factor theorem and the n zeros theorem. Now

$$p'(x)=\left[(x-r)^m\cdot q(x)\right]'=(x-r)^m\cdot(q(x))'+q(x)\cdot\left[(x-r)^m\right]'=(x-r)^m\cdot q'(x)+q(x)\cdot m\cdot(x-r)^{m-1}=$$
$$(x-r)^{m-1}\left[(x-r)\cdot q'(x)+m\cdot q(x)\right].$$

Observe that we utilized the formula for the derivative of a product and observe that $x-r$ is a common factor that is shared by both $p\ and\ p'.$

Conversely, assume $p\ and\ p'$ possess a non-trivial common factor and thus conclude that p has a multiple root. We prove the contrapositive by assuming that p does not have a multiple root where r is a root of $p(x).$ Then $p(x)=(x-r)\cdot q(x)\Rightarrow p'(x)=(x-r)\cdot q'(x)+q(x)$ so that $p\ and\ p'$ do not share a non-trivial common factor.

Our final application furnishes a very simple but often not covered application of the general quadratic function $f(x)=a\cdot x^2+b\cdot x+c;\ a,b,c\in\mathbb{R},\ a\neq 0.$ If a general quadratic polynomial has two distinct real zeros, then the abscissa of the vertex lies midway between them. The TI-89 can furnish a neat proof of this basic, yet essential idea that explores a neat connection between algebra and geometry. See **FIGURES 22-29:**

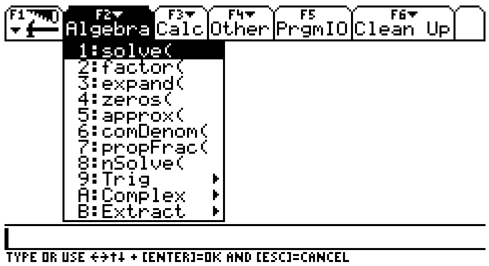


FIGURE 22: The solve command.

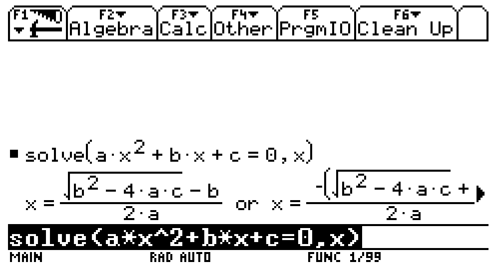
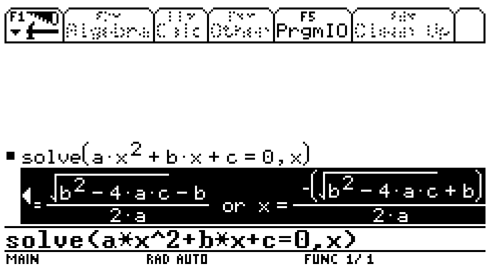


FIGURE 23: The roots of the general quadratic.



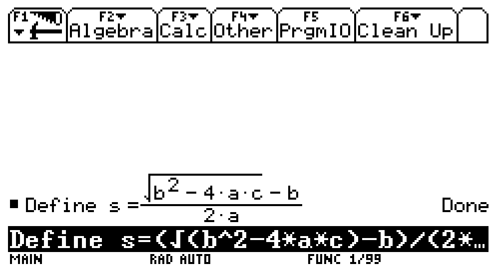


FIGURE 24: The roots of the general quadratic. FIGURE 25: Defining the first root.

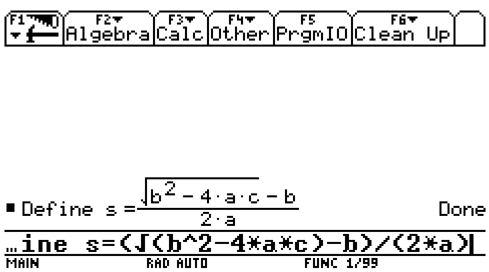


FIGURE 26: Defining the first root.

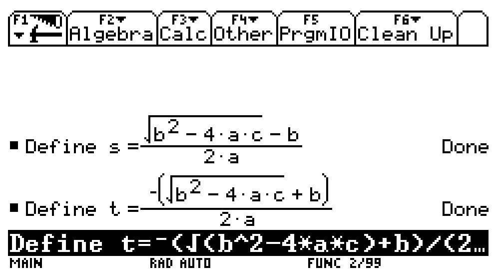
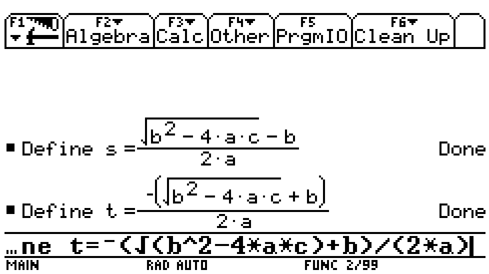


FIGURE 27: Defining the second root.



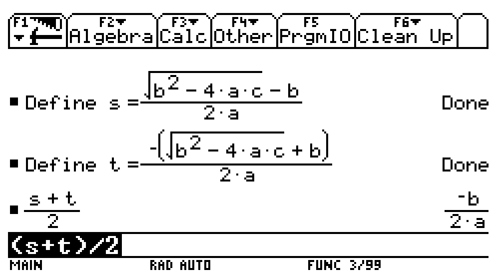


FIGURE 28: Defining the second root. FIGURE 29: The average of the two roots coincides with the abscissa of the vertex of the parabola.

Of course, it is well known that the coordinates of the vertex for the general quadratic function are $V\left(\frac{-b}{2\cdot a}, c-\frac{b^2}{4\cdot a}\right)$.

We conclude with four problems for the reader to try which serve to illustrate these ideas. The TI-89 graphing calculator can be utilized to aid in the solution to furnish an additional representation.

1. Determine the relative extrema and point of inflection for the cubic polynomial $p(x) = x^3 - 9 \cdot x^2 - 24 \cdot x + 2$. Illustrate the ideas related to the first application.

2. Consider the cubic polynomial $p(x) = x^3 - 6 \cdot x^2 + 11 \cdot x - 6$. Determine the three real zeros of this polynomial and the coordinates of the inflection point illustrating the second application.

3. Consider the cubic polynomial $p(x) = x^3 - 6 \cdot x^2 + 12 \cdot x - 8$. Illustrate the connections associated with the third application by determining the real roots of this polynomial. Also determine if the polynomial has an inflection point or a relative extremum at the zeros of this polynomial.

4. Consider the quadratic polynomial function $p(x) = 6 \cdot x^2 - 36 \cdot x + 48$. Determine the roots and the abscissa of the vertex and explain the connection to our fourth application.