

CLASS PROJECTS USING MATLAB TO ANALYZE MODELS USING IMPULSIVE DIFFERENTIAL EQUATIONS

Timothy D. Comar
Benedictine University
Department of Mathematics
5700 College Road
Lisle, IL 60532
tcomar@ben.edu

Introduction

The highlight of the second semester biocalculus course at Benedictine University is the Extended Course Project. This project requires students to read a research paper in mathematical biology, implement the model in the paper using computational software, write an expository paper on the model and its implementation, and give an external presentation on their work. The mathematical models involved in these projects may involve systems of differential equations, systems of difference equations, and impulsive differential equations. The mathematical content involves the study of the dynamics of these models including finding equilibria, studying stability, and bifurcation behavior. The project exemplifies the primary goals of the course to integrate mathematical, biological, and computational content and preparing students to engage in future undergraduate research projects.

To show how MATLAB was used in the projects, we highlight a model for a two stage-structured population models with birth pulses due to Tang and Chen (Tang and Chen, 2002), upon which a student based her project. This model uses impulsive differentiation equations. For the project, the dynamics of the model is analyzed using underlying discrete dynamical systems induced by the impulsive differential equations, called the stroboscopic map. The model exhibits interesting and complex dynamical behavior, which we illustrate with a bifurcation diagram. In this paper, we provide the MATLAB code needed to plot numerical solutions and bifurcation diagrams for this model.

Background

Before describing the particular project, we provide some background of the environment in which this project is assigned. Benedictine University offers a two-semester biocalculus sequence. This sequence is designed for students majoring in the life and health sciences. For the first semester, the students have a 4 credit hour lecture course and a 1 credit hour lab course with students in the traditional calculus course

to learn how to use Maple. The lecture course focuses on biological applications, and the mathematics covers the standard material of a typical first semester calculus course with additional material on basic difference equations and models involving discrete dynamics systems.

In the second course, which is 4 credit hour experience, the students learn basic integration techniques followed by mathematics that will help them analyze biological models given by either continuous or discrete dynamical systems. Student learn the basic dynamics of single ordinary differential equations with some attention to bifurcation behavior. The focus then turns to developing the matrix algebra and multivariable calculus needed to analyze dynamical systems involving systems of ordinary differential equations and systems of difference equations. Students work on a series of eight to ten small application-oriented laboratory projects using Excel, Maple, and MATLAB. It is in this course that the Extended Project is assigned. The basic requirements of the Extended Project are to read a journal article in mathematical biology, implement the articles's model using one of the computational software packages introduced in class, and then give both oral and written presentations. Students usually present their work in student sessions at local and regional professional meetings.

We recall the construct of impulsive differential equations that is addressed in this paper. An *implicit differential equation* is given by a system of the form:

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= f(t, \mathbf{x}), \quad t \neq mT, m \in \mathbf{Z} \\ \Delta\mathbf{x} &= A\mathbf{x}, \quad t = mT, m \in \mathbf{Z},\end{aligned}$$

where $\mathbf{x} : \mathbf{R} \rightarrow \mathbf{R}^n$, A is an $n \times n$ matrix, and T is the time period.

What this system describes is continuous behavior over the intervals $(mT, (m+1)T]$ followed by an instantaneous pulse at time $(m+1)T$, where m is an integer. It is necessary to characterize the behavior a solution right before a pulse and right after the pulse. We use the notation $\mathbf{x}(mT^-)$ to be the value of solution just before the pulse at time $t = mT$ and $\mathbf{x}(mT^+)$ to be the value of the solution immediately after the pulse at time $t = mT$ occurs. Using this notation, we can express $\Delta\mathbf{x}$ as:

$$\Delta\mathbf{x} = \mathbf{x}(mT^+) - \mathbf{x}(mT^-)$$

Impulsive differentiation can used to model a wide range of biological phenomena. Some of these include periodic vaccination or drug administration, birth pulses in a species in which births occur on a discrete, periodic schedule, and augmentation in pest management, for which more of a predatory species is added periodically. Impulsive differential equations combine features of both continuous and discrete

systems. By “sampling” solutions immediately after the pulse is applied, we obtain a system of difference equations

$$\mathbf{x}((m + 1)T^+) = g(\mathbf{x}(mT^+)).$$

This system of difference equations is called the *stroboscopic map*. Equilibria of the stroboscopic map correspond to cycles (periodic solutions) in the system. Equilibria of the stroboscopic map correspond to cycles (periodic solutions) in the system.

Systems given by impulsive differential equations fit nicely into the second semester biocalculus course. Concepts previously studied in class that arise the study of the impulsive differential equations include linear and nonlinear systems of differential equations, systems of difference equations, linearization and stability, and the basics of bifurcation theory. The project enables the students to see these concepts in practice and use the software to help with analysis of the dynamics of the systems.

Using MATLAB for Impulsive Differential Equations

To illustrate the MATLAB implementation, we focus on a simple impulsive model due to Tang and Chen (Tang and Chen, 2002). This is an age-structured model with birth pulses. We use x and y to represent the population densities of the immature and mature populations of the species, respectively. In this model, the death rate constant is d , and the maturity rate constant is δ . The meaning of the constant δ is that the mean length of the time of an individual spent in the immature class is $\frac{1}{\delta}$. The expression $be^{-(x(m^-)+y(m^-))}$ is the birth rate function, which is the average number of individuals born to an individual in the mature class, where b is a growth rate parameter. The equations for the model are given by:

$$\begin{aligned} \frac{dx}{dt} &= -dx(t) - \delta x(t) \\ \frac{dy}{dt} &= \delta x(t) - dy(t), \quad t \in (m, m + 1], m \in \mathbf{Z} \\ x(m^+) &= x(m^-) + be^{-(x(m^-)+y(m^-))}y(m^-), \quad m \in \mathbf{Z} \end{aligned}$$

The first MATLAB file we need is the file `sys1.m`, which gives the differential equations. Notice that this file does not incorporate the pulse behavior. We incorporate this when we use a numerical solving routine. The code for this file is given below.

```
%sys1
function ydot=sys1(t,x)
global d delta;
ydot(1)= -d*x(1)-delta*x(1);
```

```

ydot(2)=delta*x(1)-d*x(2);
ydot=[ydot(1);ydot(2)];

```

The next file is the numerical solving routine that we called `ode15sp.m`. This file uses the MATLAB stiff ordinary differential equation solver `ode15` for each interval and then applies the pulse behavior. As given here, the pulse is not applied after the model is numerically solved over the final interval. This model returns the numerically numerical solution to the differential equation, y , along with the times values, given by t , at which the systems is evaluated. As a byproduct, we store the solution to the corresponding stroboscopic map in the global variable z .

```

%ode15sp
function [t,y]=ode15sp(ti,tf,y10, y20,b,pulsestep)
global d delta z;
clear y;clear z;y(1,1)=y10; y(1,2)=y20;
rep=ceil((tf-ti)/pulsestep); clear s ytemp;
z= zeros(rep,2);z(1,1)=y10;z(1,2)=y20;
%rep is the number of intervals of length pulsestep that over which the
%system is evaluated.
%evalpts is the number of points at which times the systems is
%evaluated over an interval
%the evaluation over the first interval
%z contains the points generated by the stroboscopic map.
[s,ytemp]=ode15s('sys1',[ti, ti+pulsestep],[y10, y20]);
[evalpts, cols]=size(ytemp);
for j=1:evalpts
    t(j)=s(j);
    y(j,1)=ytemp(j,1);
    y(j,2)=ytemp(j,2);
end;
%the evaluation over the succeeding intervals
for k=1:rep-1
    [sizey,cols]=size(y);
    y1int=y(sizey,1)+ b*exp(-y(sizey,1) -y(sizey,2))*y(sizey,2);
    y2int=y(sizey,2);
    z(k+1,1)=y1int;z(k+1,2)=y2int;
    clear s ytemp;
    [s,ytemp]=ode15s('sys1',[ti+k*pulsestep,ti+(k+1)*pulsestep],[y1int,
y2int]);
    [evalpts, cols]=size(ytemp);
    for j=1:evalpts
        t(sizey+j)=s(j);

```

```

        y(sizey+j,1)=ytemp(j,1);
        y(sizey+j,2)=ytemp(j,2);
    end;
end;

```

We are able to plots numerically generated solutions to the system. We do this with the file `solutionplot.m`. This file plots the x and y population density times series as well as the solution in the xy -phase plane. For example, we plot a solution with $d = 0.2$, $\delta = 0.4$, $b = 100$. Note that the time step between pulses is 1. Initial values are chosen to be $x_0 = 1.5$ and $y_0 = 4.24$. In Figure 1, we see the plot of the x time series, and we see the solution plot in the xy -phase plane in Figure 2. These figures indicate that this solution illustrates periodic behavior.

```

%solutionplot
figure(1);clf(1);figure(2);clf(2);figure(3);clf(3);
global d delta z;
d=0.2;delta=0.4;
[t,y]=ode15sp(0,1000,1.5,4.24,100,1);
[sizey,cols]=size(y);
%plot of immature class x
figure(1);plot(t(12610:sizey),y(12610:sizey,1),'r-');
xlabel('t');ylabel('x(t)');
%plot of immature class x
figure(2);plot(t,y(:,2),'b-');xlabel('t');ylabel('y(t)');
%plot of solution in the xy-phase plane
figure(3);plot(y(12610:sizey,1),y(12610:sizey,2),'r-');
xlabel('x(t)');ylabel('y(t)');

```

To better understand the complete picture of the dynamics of the system, it is often helpful to analyze the bifurcation diagram of the stroboscopic map of the system. Recall we generated the stroboscopic map using the file `ode15sp` and stored it in the global variable `z`. Probably the easiest method for obtaining the stroboscopic map is by embedding its generation in the numerical solving routine as we have done. This method is particular useful when it is not possible to obtain manageable formula for the stroboscopic map. For our example, there is a nice form for the stroboscopic map, which is given by:

$$\begin{aligned}
 x_{m+1} &= x_m e^{-(\delta+d)} + b[y_m + x_m(1 - e^{-\delta})]e^{-d+e^{-d}(x_m+y_m)} \\
 y_{m+1} &= e^{-d}(1 - e^{-\delta})x_m + e^{-d}y_m,
 \end{aligned}$$

where m is a nonnegative integer.

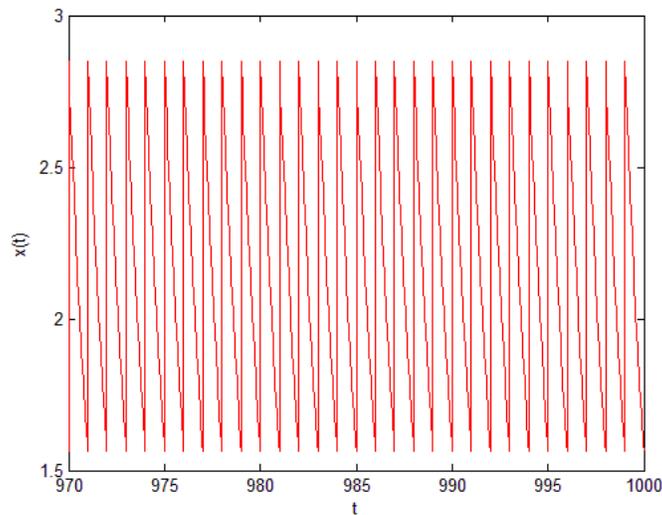


Figure 1: The x times series for the solution with $d = 0.2$, $\delta = 0.4$, $b = 100$ and initial values $x_0 = 1.5$ and $y_0 = 4.24$

The code which generates the stroboscopic map and a bifurcation diagram is given in the file `bif.m`. For this particular bifurcation diagram the sum $x_m + y_m$ as b varies from 500 to 1,002,000. Again, we use $d = 0.2$ and $\delta = 0.4$. For each value of b , a sequence of 1000 terms is generated, and then the final 200 terms are plotted in the bifurcation diagram. We see initial periodic behavior for small values of b followed by period doubling, chaotic behavior, and periodic windows appearing within the chaotic areas. The dynamics of this system are indeed complex. The plot of the bifurcation diagram is given in Figure 3.

```
%bif.m
figure(1);clf(1);
global d delta;
d = 0.2; delta = 0.4;
clear z;z=zeros(1000,2);
z(1,1)=1.5;z(1,2)=4.24;
Max=501; step=2000; interval=Max*step;
for n=1:Max
    b=step*n;
    for k=1:999
        z(k+1,1)=z(k,1)*exp(-d - delta) + b*(z(k,2) +
            z(k,1)*(1-exp(-delta)))*exp(-(d+ exp(-d)*(z(k,1)+z(k,2)))));
        z(k+1,2)=exp(-d)*(1-exp(-delta))*z(k,1)+exp(-d)*z(k,2);
    end
end
```

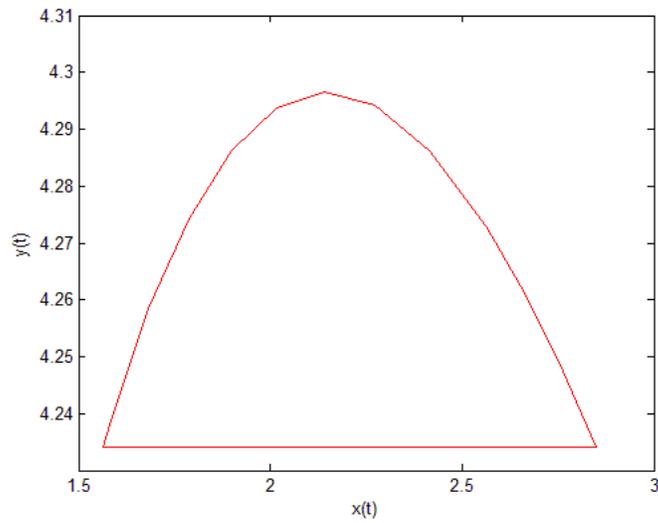


Figure 2: The solution plot in the xy -phase plane for the solution with $d = 0.2$, $\delta = 0.4$, $b = 100$ and initial values $x_0 = 1.5$ and $y_0 = 4.24$

```

plot(b*ones(200),z(801:1000,1)+z(801:1000,2),'.','MarkerSize',10);
xlabel('b');ylabel('x(n)+y(n)');
hold on
z(1,1)=1.5;z(1,2)=4.24;
end

```

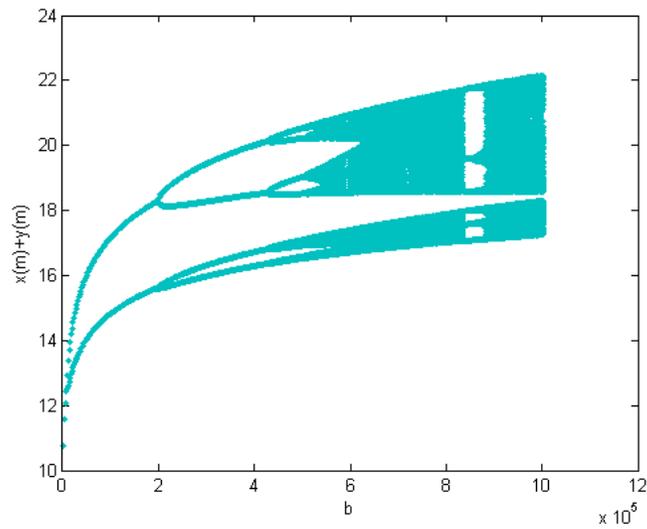


Figure 3: The bifurcation diagram for $d = 0.2$ and $\delta = 0.4$ as b varies from 2000 to 1,002,000

Acknowledgments

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References

- [1] S. Tang and L. Chen, 2002, "Density-Dependent birth rate, birth pulses and their population dynamic consequences," *Journal of Mathematical Biology*, **44**, 185-199.