EXPLORING THE FIBONACCI SEQUENCE OF ORDER TWO WITH CAS TECHNOLOGY

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ABSTRACT: The sequence recursively defined by $P_1 = 1,\ P_2 = 2,\ and\ P_n = 2 \cdot P_{n-1} + P_{n-2}\ for\ n \geq 3$ is often called the Pell Sequence or Fibonacci sequence of order two. In this paper, the TI-89 and MATHEMATICA 8.0 enable one to examine ratios of successive terms, explore divisibility and periodicity, furnish early prime outputs and provide algebraic proofs of palatable number tricks.

We begin by considering the sequence 1, 2, 5, 12, 29, 70, 169, 408, 985,.... This sequence is often referred to as the Pell sequence or a Fibonacci sequence of order two. Observe that the initial two terms are fixed at 1 and 2. After that, each term is the sum of twice the previous term plus the term two places before it. To cite three early examples,

$$P_3 = 2 \cdot P_{3-1} + P_{3-2} = 2 \cdot P_2 + P_1 = 2 \cdot 2 + 1 = 4 + 1 = 5,$$
$$P_4 = 2 \cdot P_{4-1} + P_{4-2} = 2 \cdot P_3 + P_2 = 2 \cdot 5 + 2 = 10 + 2 = 12\ and$$
$$P_5 = 2 \cdot P_{5-1} + P_{5-2} = 2 \cdot P_4 + P_3 = 2 \cdot 12 + 5 = 24 + 5 = 29.$$

The table below furnishes the initial twenty terms in the sequence:

n:	P_n:	n:	P_n:
1	1	11	5741
2	2	12	13860
3	5	13	33461
4	12	14	80782
5	29	15	195025
6	70	16	470832
7	169	17	1136689
8	408	18	2744210
9	985	19	6625109
10	2378	20	15994428

Examining the table in some detail for this small data set, one is led to the following conjectures concerning divisibility in this sequence:

Conjecture 1: Every even numbered term is even.
Conjecture 2: Every fourth term is divisible by both 3 and 4.

Conjecture 3: Every third term is divisible by 5.
Conjecture 4: Every sixth term is divisible by 7.
Conjecture 5: Every seventh term is divisible by 13.

The proofs of these conjectures are easily resolved via The Principle of Mathematical Induction. Let us illustrate the proof of **Conjecture 3:**

Let $P(n)$ *be the proposition* $5 \mid P_{3 \cdot n}$.

Step 1: *Prove* $P(1)$ *is true (This is the Basis Case)*:

$5 \mid P_{3 \cdot 1} \Leftrightarrow 5 \mid P_3 \Leftrightarrow 5 \mid 5 \Leftrightarrow 5 = 5 \cdot 1$ *which is true.*

Step 2: *Assume* $P(k)$ *is true (This is the Induction Hypothesis)*:

We assume $5 \mid P_{3k}$.

Step 3: *Prove that* $P(k) \Rightarrow P(k+1)$. *(This establishes the induction)*:

We prove $5 \mid P_{3k} \Rightarrow 5 \mid P_{3(k+1)} \Leftrightarrow 5 \mid P_{3k} \Rightarrow 5 \mid P_{3 \cdot k+3}$. *Using the recursion relation in our sequence we observe that* $P_{3 \cdot k+3} = 2 \cdot P_{3 \cdot k+2} + P_{3 \cdot k+1} = 2 \cdot (2 \cdot P_{3 \cdot k+1} + P_{3 \cdot k}) + P_{3 \cdot k+1} = 4 \cdot P_{3 \cdot k+1} + 2 \cdot P_{3 \cdot k} + P_{3 \cdot k+1} = 5 \cdot P_{3 \cdot k+1} + 2 \cdot P_{3 \cdot k}$. *By the Induction Hypothesis*, $5 \mid P_{3k} \Rightarrow 5 \mid 2 \cdot P_{3 \cdot k}$.

In addition, $5 \mid 5 \Rightarrow 5 \mid 5 \cdot P_{3 \cdot k+1}$. *Since* $5 \mid 2 \cdot P_{3 \cdot k}$ *and* $5 \mid 5 \cdot P_{3 \cdot k+1} \Rightarrow 5 \mid [5 \cdot P_{3 \cdot k+1} + 2 \cdot P_{3 \cdot k}] \Leftrightarrow 5 \mid P_{3 \cdot k+3} \Leftrightarrow 5 \mid P_{3(k+1)}$ *establishing our induction.*

Step 4: *By The Principle of Mathematical Induction, since the statement is true for* $n=1$, *it must be true for* $n = 1+1 = 2$, $n = 2+1 = 3, \ldots$ *i.e.* $\forall n \in \mathbb{N}$. *This completes the proof of our contention.*

The TI-89 illustrates a graph, Table, and Trace Values. Divisibility is achieved using the mod command. Consider the following screen captures in **FIGURES 1-16** where the calculator is placed in SEQUENCE MODE:

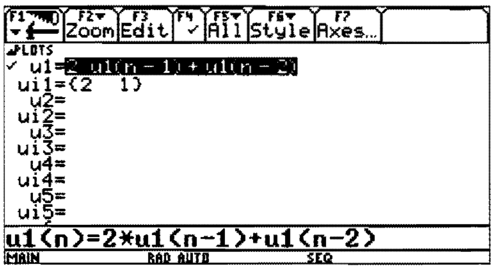

FIGURE 1: The Sequence Input.

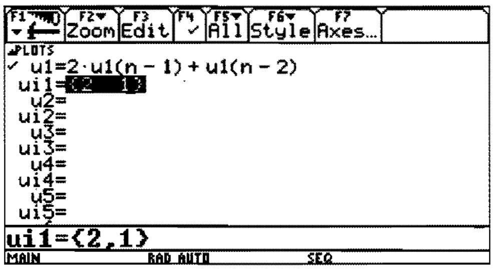

FIGURE 2: The Sequence Input.

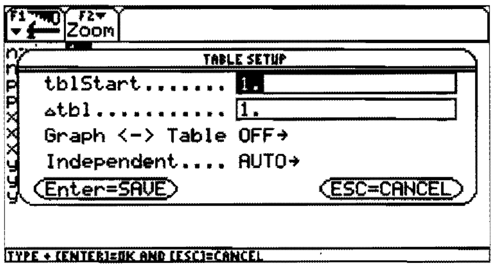

FIGURE 3: The Table Setup.

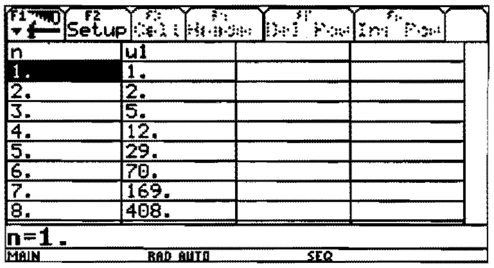

FIGURE 4: The Table Revealed.

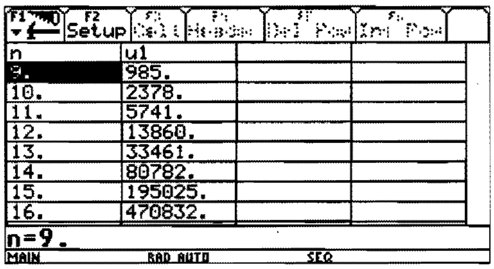

FIGURE 5: The Table Revealed.

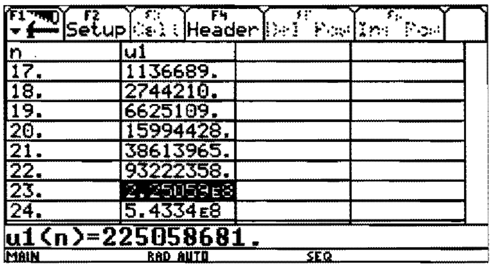

FIGURE 6: The Table Revealed.

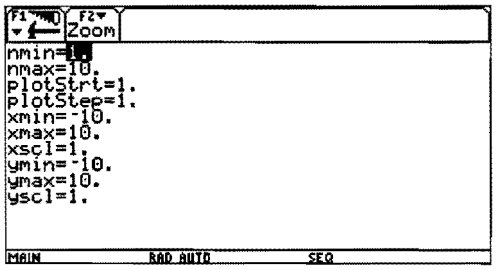

FIGURE 7: The Standard Window.

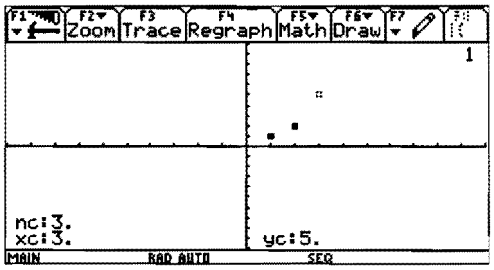

FIGURE 8: The Third Term is 5.

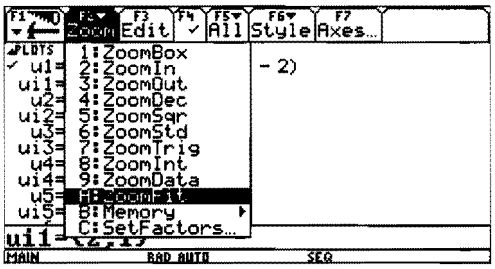

FIGURE 9: The Zoom Fit Option.

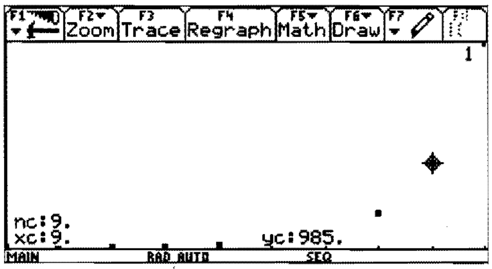

FIGURE 10: The Ninth Term is 985.

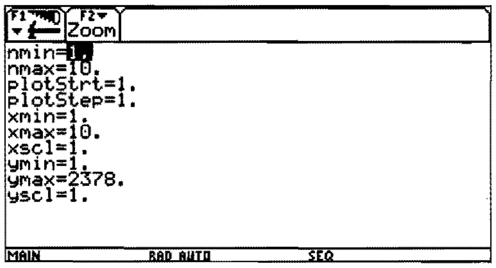

FIGURE 11: The Zoom Fit Window.

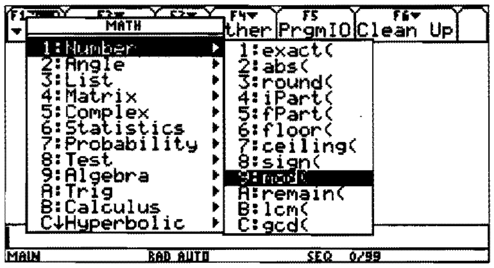

FIGURE 12: The Mod Key.

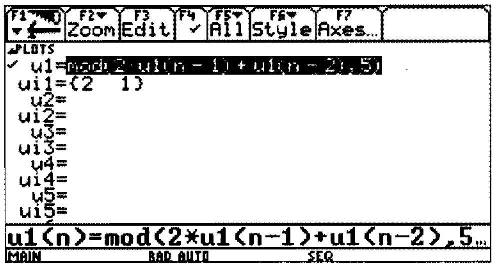

FIGURE 13: The Sequence Input.

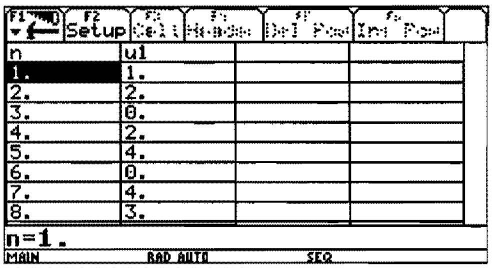

FIGURE 14: The Table Revealed.

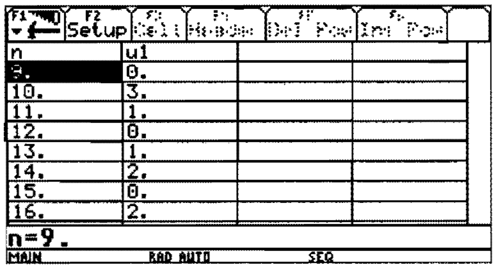

FIGURE 15: The Table Revealed.

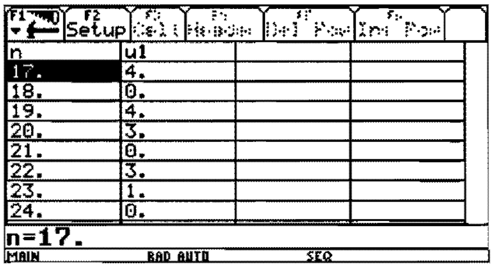

FIGURE 16: The Table Revealed.

FIGURES 1-2 input the Pell Sequence, **FIGURE 3** is the Table Setup, **FIGURES 4-6** generate the initial two dozen terms of the sequence, **FIGURE 7** is the Standard Viewing Window for the sequence, **FIGURE 8** is the calculator generated graph with the third term of the sequence displayed (5), **FIGURE 9** is the Zoom Fit Option, **FIGURE 10** fits the first ten data points to the adjusted window depicted in **FIGURE 11, FIGURE 12** illustrates the mod command, **FIGURE 13** is the input to see which terms of the sequence might by divisible by five, and **FIGURES 14-16** give rise to our conjecture that every third term in the sequence is divisible by five, a fact we proved by mathematical induction earlier. **FIGURES 14-15** also shows the periodicity of the Pell Sequence modulo 5 (which is of length one dozen) with the sequence of remainders being 1, 2, 0, 2, 4, 0, 4, 3, 0, 3, 1, 0, 1, 2, 0, 2, 4, 0, 4, 3, 0, 3, 3, 1, 0, ….

Our first table illustrates the initial time the first thirty prime numbers enter the Pell sequence.

Pr*ime*	*Term*	*Output Value*
2	P_2	2
3	P_4	12
5	P_3	5
7	P_6	70
11	P_{12}	13860
13	P_7	169
17	P_8	408

19	P_{20}	15994428
23	P_{22}	93222358
29	P_5	29
31	P_{30}	107578520350
37	P_{19}	6625109
41	P_{10}	2378
43	P_{44}	24580185800219268
47	P_{46}	143263821649299118
53	P_{27}	7645370045
59	P_{20}	15994428
61	P_{31}	259717522849
67	P_{68}	37774750930342781945186500
71	P_{70}	220167382952941249990598278
73	P_{36}	21300003689580
79	P_{26}	3166815962
83	P_{84}	50305164660422142002238655969020
89	P_{44}	24580185800219268
97	P_{48}	835002744095575440
101	P_{51}	11749380235262596085
103	P_{34}	3654502875938
107	P_{108}	77308816174220163766296465781233402364740
109	P_{110}	45058880117184178512984722732832363869422
113	P_{28}	18457556052

Our second table furnishes a list of the values of n yielding the prime outputs in the Pell sequence P_n for $n \leq 500$:

P_n is prime for $n = 2, 3, 5, 11, 13, 29, 41, 53, 59, 89, 97, 101, 167, 181$ *and* 191.

n:	Prime Output Value P_n:
2	2
3	5
5	29
11	5741
13	33461
29	44560482149
41	1746860020068409

53	68480406462161287469
59	13558774610046711780701
89	4125636888562548868221559797461449
97	4760981394323203445293052612223893281
101	161733217200188571081311986634082331709
167	2964793555272799671946653940160950323792169332712780937764687561
181	677413820257085084326543915514677342490435733542987756429585398537901
191	4556285254333448771505063529048046595645004014152457191808671945330235841

The primality of n is necessary for P_n to be prime. The converse is not valid; for 7 is prime, but $P_7 = 169 = 13^2$ and hence composite.

We next consider the ratios of successive terms in the Pell sequence. See **FIGURES 17-19** where one examines the ratio of an even-numbered term to the preceding odd numbered term and **FIGURES 20-22** where we consider the ratio of an odd numbered term to the preceding even numbered term.

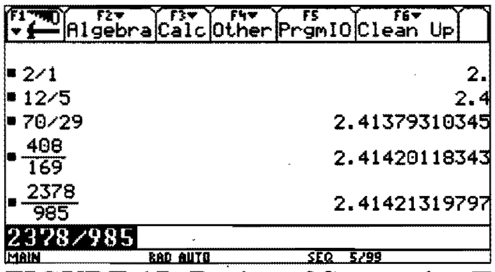

FIGURE 17: Ratios of Successive Terms.

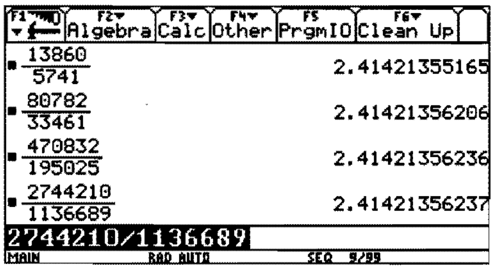

FIGURE 18: Ratios of Successive Terms.

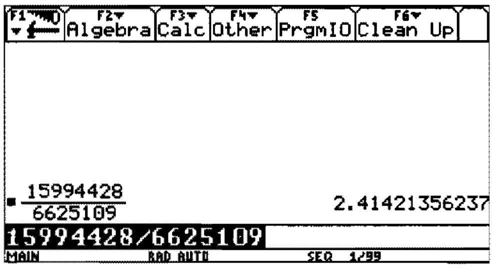

FIGURE 19: Ratios of Successive Terms.

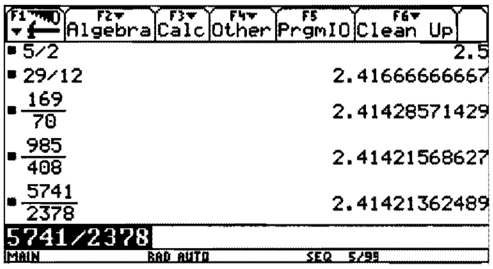

FIGURE 20: Ratios of Successive Terms.

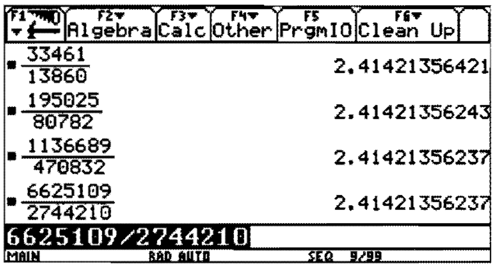

FIGURE 21: Ratios of Successive Terms.

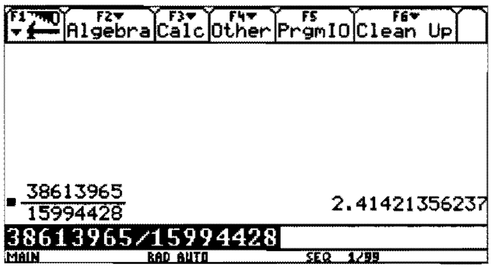

FIGURE 22: Ratios of Successive Terms.

The sequence in **FIGURES 17-19** is clearly increasing while the sequence in **FIGURES 20-22** is clearly decreasing. Both sequences are approaching the same number; namely $1+\sqrt{2}$. This is the Golden Mean analogue of the number $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.61803398875$ in the Fibonacci sequence. This Pell sequence constant satisfies the quadratic equation $x^2 = 2\cdot x + 1$ in much the same manner that the constant $\Phi = \frac{1+\sqrt{5}}{2}$ satisfies the quadratic equation $x^2 = x+1$ in dealing with the Fibonacci sequence. While the n-th Fibonacci number F_n satisfies the Binet Formula $F_n = \left(\frac{1}{\sqrt{5}}\right)\cdot\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$, the n-th Pell number P_n satisfies the Binet-like formula

$P_n = \left(\frac{1}{2\cdot\sqrt{2}}\right)\cdot\left[\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right]$. See **FIGURES 23-24:**

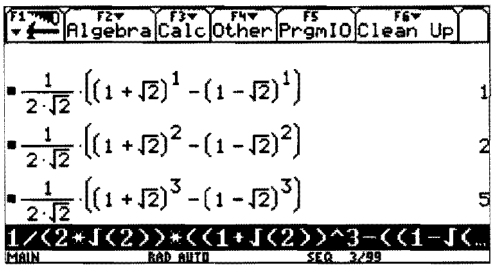

FIGURE 23: Explicit Closed Formula.

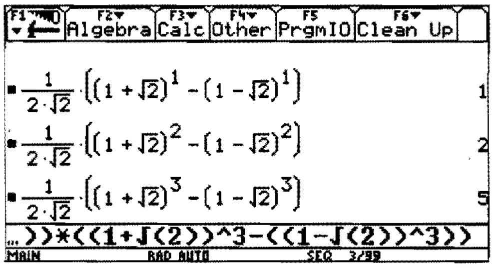

FIGURE 24: Explicit Closed Formula.

Our concluding activities focus on some interesting number tricks associated with this sequence as well as securing the prime factorizations of the first twenty-five Pell numbers.

Consider the sum of any four, eight, and twelve consecutive terms in the Pell sequence. Let the first two terms be x *and* y respectively. The next terms can be secured by the TI-89 as in **FIGURES 25-27:**

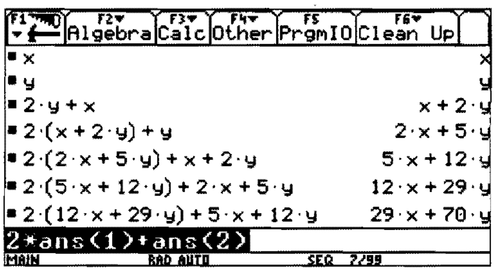

FIGURE 25: The First Seven Terms.

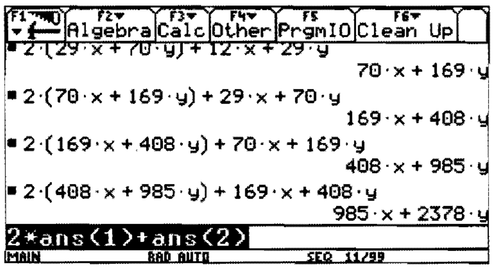

FIGURE 26: Terms Eight To Eleven.

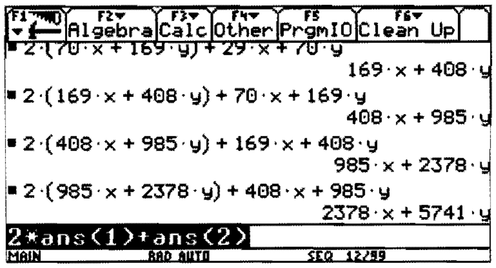

FIGURE 27: The Twelfth Term.

Note that the coefficients of the terms are all Pell numbers. Now the sum of four consecutive terms is

$x+y+x+2\cdot y+2\cdot x+5\cdot y=4\cdot x+8\cdot y$. *Note that* $\frac{4\cdot x+8\cdot y}{4}=x+2\cdot y$, *the third term in the sequence.*

The sum of four consecutive terms in the Pell sequence is divisible by four and the quotient is the third term in the sequence. See **FIGURE 28** where the calculator furnishes a proof:

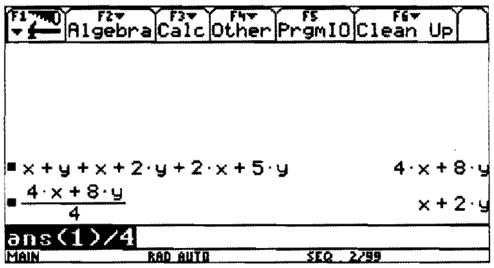

FIGURE 28: A Calculator Proof.

Next examine the sum of eight consecutive terms in the sequence and divide the sum by twenty-four. The quotient is the fifth term in the sequence. See **FIGURES 29-30:**

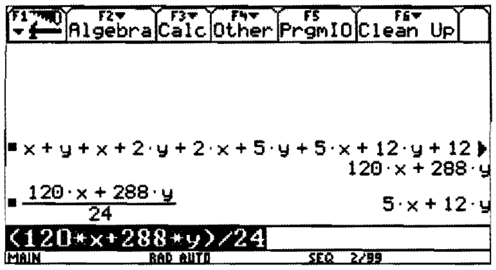

FIGURE 29: A Calculator Proof.

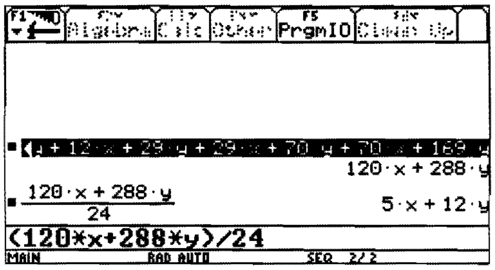

FIGURE 30: A Calculator Proof.

Finally the sum of twelve consecutive terms is divisible by 140 and the quotient is the seventh term in the sequence. See **FIGURES 31-34:**

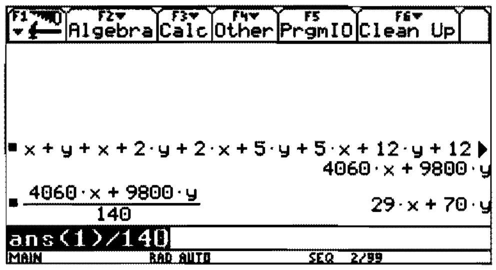

FIGURE 31: A Calculator Proof.

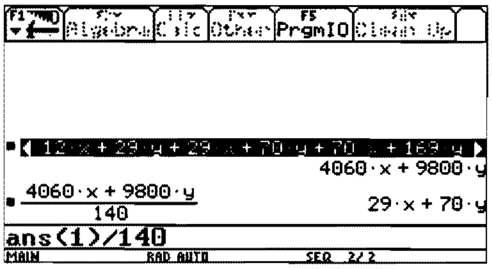

FIGURE 32: A Calculator Proof.

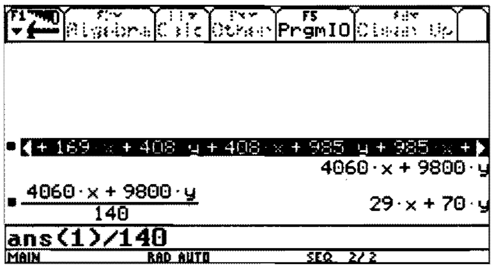

FIGURE 33: A Calculator Proof.

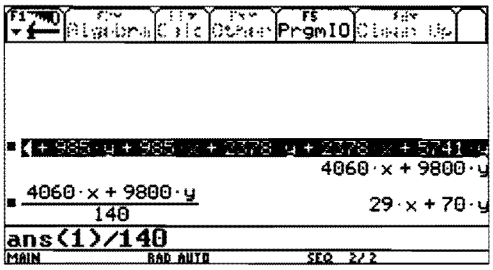

FIGURE 34: A Calculator Proof.

We conclude by securing the factorizations of the first twenty-five Pell numbers in **FIGURES 35-49** below:

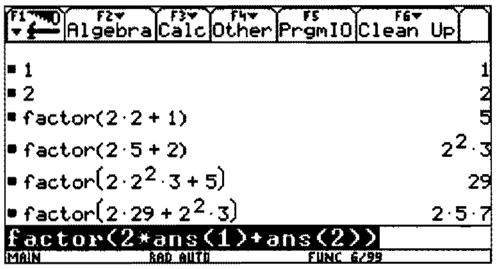

FIGURE 35: The Prime Factorizations.

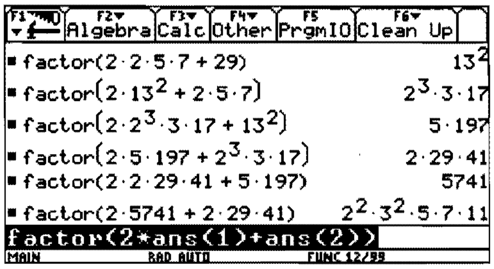

FIGURE 36: The Prime Factorizations.

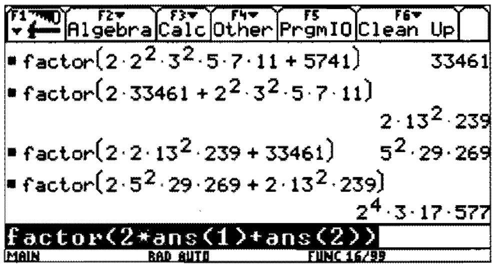

FIGURE 37: The Prime Factorizations.

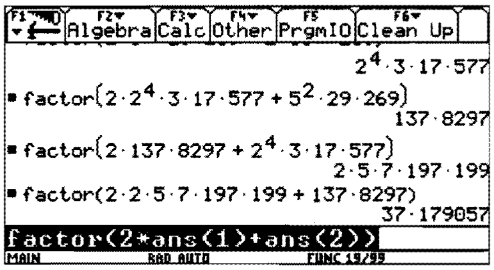

FIGURE 38: The Prime Factorizations.

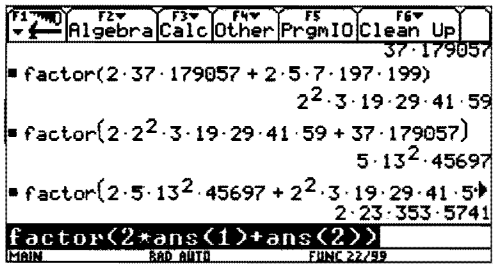

FIGURE 39: The Prime Factorizations.

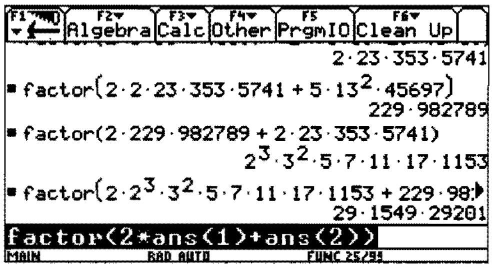

FIGURE 40: The Prime Factorizations.

Conclusion: Fibonacci-like sequences are very amenable for discovering neat insights that have broadly based appeal. This paper explored some elementary ideas with the Fibonacci sequence of order two also referred to as the Pell sequence in the literature. Applications of this sequence abound. One may extend the ideas extrapolated in this paper by forming new conjectures associated with this and other Fibonacci-like sequences. For example, Neil J.A. Sloane, the founder of the excellent OEIS (The On-Line Encyclopedia of Integer Sequences) presents the initial five hundred terms in this sequence. I have utilized MATHEMATICA 8.0 to secure the prime factorizations of all but one of the initial three hundred members of this sequence; namely the two hundred ninety-ninth which is currently in progress. A very neat application to geometry is manifested in the Pell sequence with regards to right triangles commonly known as Theon's Ladder which is the subject of a stimulating article in the *College Mathematics Journal* co-authored by a colleague of mine with one of his top students.

Reference:

1. Thomas J. Osler (with Shaun Giberson), *Extending Theon's Ladder to any Square Root,* The College Mathematics Journal, 35 (2004), pp. 222-226.