TOWARDS A PEDAGOGICAL CAS

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There was an article in the *Technology Tips* section of the *Mathematics Teacher* journal from the National Council of Teachers of Mathematics (Volume 102, Number 6, February 2009) titled *TI-Nspire Calculators: Better but Still Not Perfect* (by Michael J. Bosse and Kwaku Adu-Gyamfi) that discussed the state of the art of CAS calculators for the teaching of mathematics. Specifically, the article addressed two functional areas: 1) Graphing and 2) Automatically Rewriting Expressions as being important areas to improve. The section on automatic simplification closed with the observation that a CAS for the teaching of mathematics should separate entry and operations on expressions. To these categories, we add a third: 3) Step-by-Step algorithms. In this workshop, the author sets out the capabilities of the HP 40gs CAS calculator, based on these 3 categories:

1. Graphing Fidelity
2. Separating Entry of an Expression and Operating on Expressions
3. Working Step by Step

We believe that improvements in these areas lead towards a pedagogical CAS.

Graphing Fidelity

The graphing section of the article contains a total of 7 examples, numbered (1) through (7), which the authors explain are incorrectly graphed by at least one graphing calculator. Of these 7, 5 are now correctly graphed by other graphing calculators. Let us examine the graphs produced by the HP 40gs.

The first example is

(1) $y1 = \left(\sqrt{x}\right)^2 \neq y2 = x$

The desired graphing behavior is to respect the different domains and restrict *y1* to non-negative real numbers. The function *y2* has no such restriction. Figure 1 shows *y1* entered as *f1(x)* on the HP 40gs, while Figure 2 shows the correct graph.

We prefer the more exact notation of *f1(x)* to the somewhat ambiguous *y1(x)*.

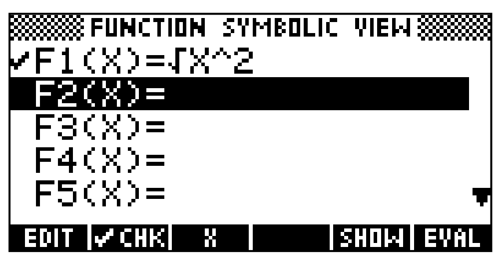

Figure 1

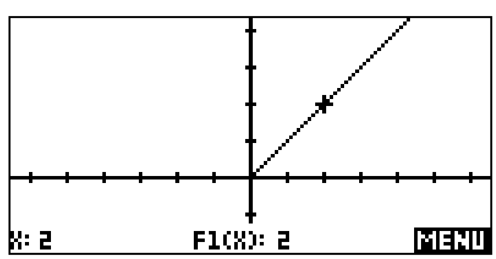

Figure 2

The second example is

$$(2) \quad y1 = \frac{\sqrt{x-1}}{\sqrt{x-3}} \neq y2 = \sqrt{\frac{x-1}{x-3}}$$

In this example, the domain of *y1* is restricted to real numbers greater than 3, while *y2* is not defined for x where $1 \le x < 3$.

Figure 4 and Figure 5 show the input and correct graphing behavior for *y1*.

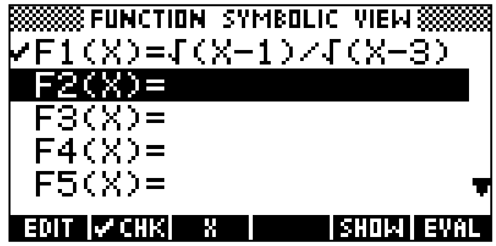

Figure 4

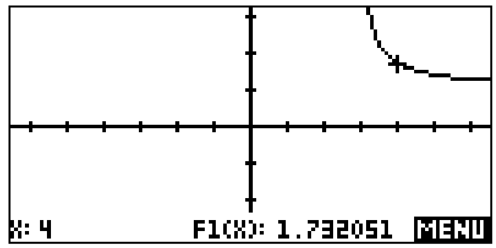

Figure 5

Although it is not clear from Figure 5, if one traces to x-values of 3 or less, the tracer readout shows that *f1(x)* is undefined for these values, as required.

Figure 6 and Figure 7 show the input and correct graphing behavior (respectively) for *y2* as well.

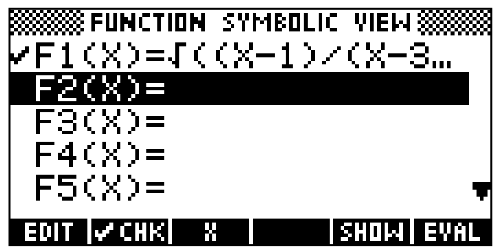

Figure 6

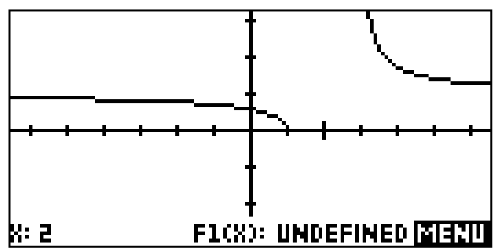

Figure 7

The third example is

$$(3) \quad y1 = \sqrt{x-3}\sqrt{1-x} \neq y2 = \sqrt{(x-3)(1-x)}$$

The use of *y1* and *y2* are continued in this example for consistency and clarity, although omitted by the authors.

In this example, *y1* is not defined for any real numbers and thus no graph should be produced. The correct graphing behavior in this case is exhibited by the HP 40gs, though the non-graph is not shown here.

The domain of *y2* is limited to x such that $1 \le x \le 3$. Figure 8 shows *y2* entered into the HP 40gs as *f1(x)*.

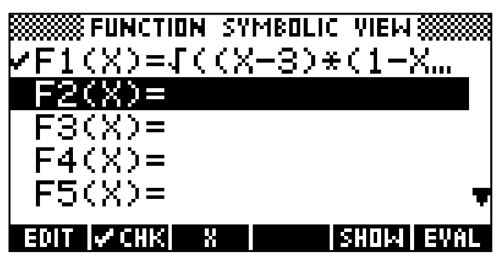

Figure 8

Figure 9 show the required graphing behavior. As with the previous examples, tracing to x-values outside of the domain clearly indicate that *f1(x)* is undefined for those values.

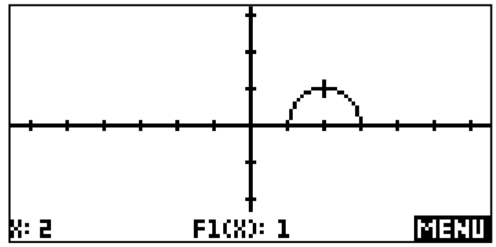

Figure 9

To return to our graphing examples, Examples 4, 6, and 7 illustrate cases where y1 is not defined for any real numbers, while y2 is. In these cases, the HP 40gs performs as desired, though the (non)graphs are not provided here. For reference, the equations are:

$$(4)\quad y1 = \ln(1-x) - \ln(x-2) \neq y2 = \ln\left(\frac{1-x}{x-2}\right)$$

$$(6)\quad y = \frac{2\sin^{-1} x + \pi}{\sqrt{x-2}}$$

$$(7)\quad y = i^{\sqrt{x}}$$

Again, we use *y1* and *y2* for consistency here, although the article omits them in these examples.

Example 5 is

$$(5)\quad y1 = \frac{(-4)^x}{(-2)^x} \neq y2 = \left(\frac{-4}{-2}\right)^x = 2^x$$

The authors note that y1 may graph incorrectly if the plot style is chosen to be continuous. Figure 10 shows y1 entered as f1(x) with the Connected option on (Figure 11). Note that the HP 40gs considers f1(x) to be defined for all integer values of x (Figure 12) and undefined for all other real numbers (Figure 13).

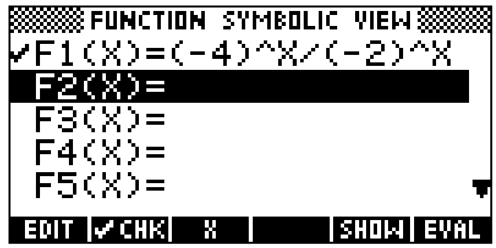

Figure 10

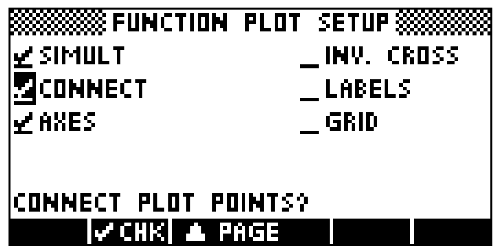

Figure 11

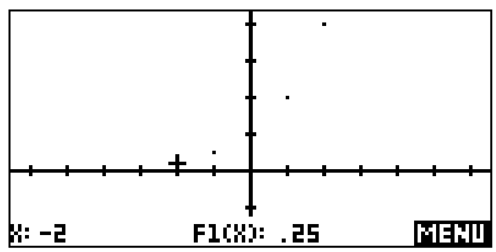

Figure 12

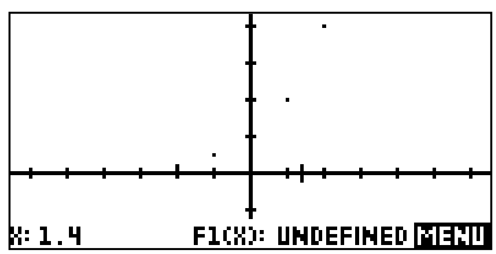

Figure 13

It seems that the HP 40gs correctly handles all 7 examples brought forth in the article.

Separating Entry of an Expression and Operating on Expressions

In the second section of their article, Bosse and Adu-GYamfi provide another set of examples. The underlying point of these examples is to show that CAS calculators automatically perform simplifications without explicit direction from the user. In their recommendations, the authors make the following argument:

> In this discussion, we continue the argument found in Bosse and Nandakumar (2004) that loading any expression into the calculator should be differentiated from performing an operation on an expression. The user should be allowed to enter any expression onto the screen and have nothing occur until the calculator is prompted to perform some operation.

To be fair, the HP 40gs handles most, if not all, of the examples in approximately the same way as other CAS graphing calculators. But for the HP 40gs, the automatic simplification of expressions is directly tied to the entry of and operation upon expressions. Thus we combine the two categories in this section and proceed to an examination of the HP 40gs CAS editor. In this examination, the audience will see the steps taken by HP to address the authors' concerns.

The HP 40gs CAS editor employs what we call an *Interactive Editor*. The *Interactive Editor* performs two major functions.

1. It makes sure that the expression under edit is always valid.
2. It allows the user to operate on the expression or any of its sub-expressions.

Let us illustrate these two roles with an example from the same article, namely:

$$\frac{\sqrt{4\cdot -1}}{\sqrt{-1}}$$

Figure 14 shows the user entering the numerator. In Figure 15, the user selects the numerator and in Figure 16 invokes the operation of division. If the user presses the Enter key at this point, nothing happens because the expression is not yet valid. In Figure 17, the user has completed the entire expression. Note that the user has not pressed the Enter key and no action is being taken by the HP 40gs. At this point, the user can select the entire expression or any valid sub-expression and invoke a command.

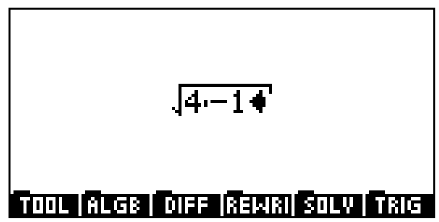

Figure 14

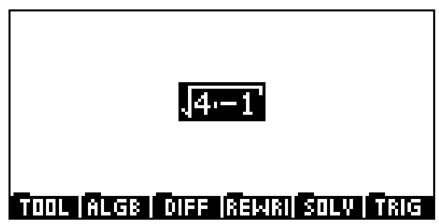

Figure 15

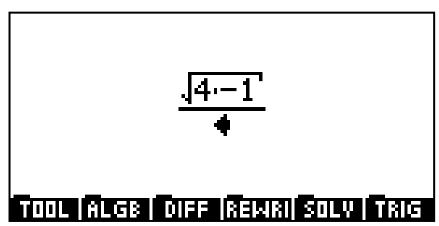

Figure 16

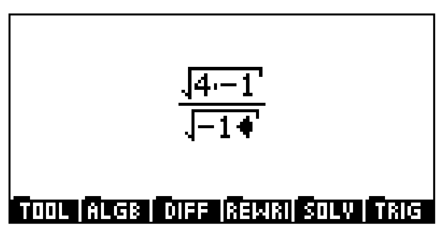

Figure 17

For example, in Figure 18, the user has selected just the radicand in the numerator. The user can press the Enter key or invoke the Simplify command to see the result: -4. Figure 19 shows the result after the Enter key is pressed.

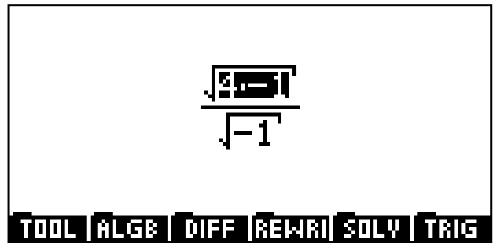

Figure 18

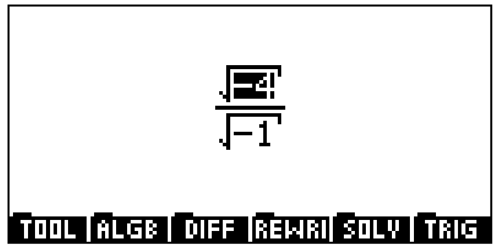

Figure 19

By default, the HP 40gs CAS operates in real versus complex mode. If an operation is requested that requires the CAS to switch to complex mode to complete, then the user will be asked if the CAS may switch to complex mode. Figure 20 shows the request after the user has selected the entire expression and pressed ENTER.

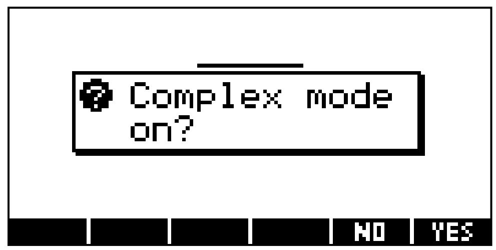

Figure 20

If the user declines to switch modes, a simplification error will be displayed and the previous expression is retained. In Figure 21, the user has declined to enter complex mode by pressing the NO key in Figure 20, resulting in the error message shown. Pressing OK at this point returns the user to the previous expression, as shown in Figure 22. If the user selects YES in Figure 20, then the complex intermediate results are accepted and the expression is simplified to 2. In the case of an expression with multiple operations, the user may be asked to switch modes multiple times.

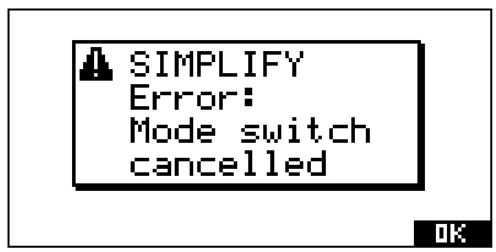

Figure 21

This approach allows the user to distinguish between contexts in which only real-number calculations are appropriate and those in which the larger realm of complex operations can be invoked.

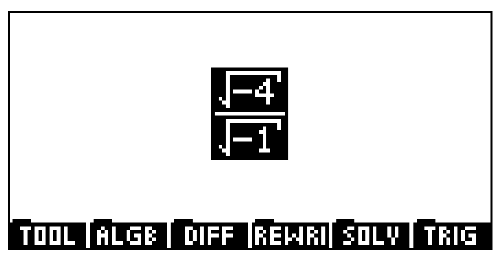

Figure 22

Another advantage of the HP 40gs is that any command can be typed in, letter by letter, from the keyboard. For example, typing S, I, N, (, π, /, 2,), and then pressing the Enter key gives the same result as using the SIN key. This is shown in Figures 23-25. The inputs and outputs are identical, regardless of mode of entry.

In summary, the HP 40gs *Interactive Editor* gives you the following benefits:

- Expressions can be entered and displayed without being simplified or operated upon
- An expression or any of its valid sub-expressions can be selected and operated upon
- You can enter commands via menus or just type them in alphabetically and the CAS treats both input methods the same

Figure 23

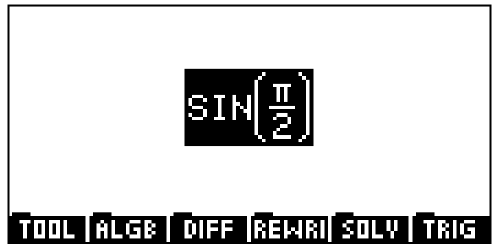

Figure 24

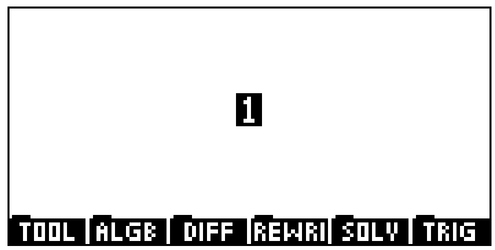

Figure 25

Working Step by Step

In this last section, we look at ways in which the automatic simplification of a CAS can be modified to make intermediate results apparent, thus tuning the CAS for more pedagogical purposes.

The HP 40gs CAS has a mode setting that allows it to differentiate step by step. Figure 26 shows the CAS Modes menu, with the Step/Step option checked to activate Step by Step mode.

In Figure 27, the expression $\sin(2x^3)$ has been entered. As discussed previously, one can now operate on this expression at will.

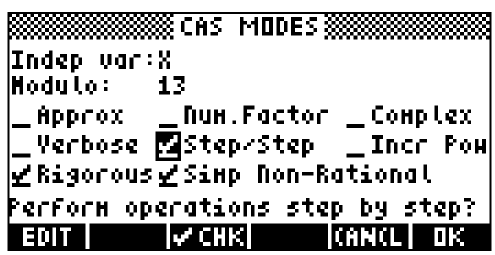

Figure 26

Figure 27

In Figure 28, the d/dx key is pressed to find the first derivative of the expression and a variable of differentiation is supplied (namely x).

Figure 29 shows the result when the ENTER key is pressed. The Chain Rule is explicitly invoked.

The final result is shown in Figure 30 after the ENTER key has been pressed again.

Figures 31 and 32 show similar results for an expression invoking the Quotient Rule.

One obtains similar results for expressions invoking the Product Rule.

Although Step by Step mode has not been implemented across the board in the HP 40gs CAS, it has been implemented in the high-value area of differentiation, where the benefits are clear.

Conclusions

In this workshop, you have experienced the following pedagogical benefits of the HP 40gs:

- The degree of graphing fidelity of the HP 40gs
- The Interactive CAS Editor that separates the entry of an expression from operations upon the expression and even allows you to operate on sub-expressions
- The Step by Step mode available for derivatives
- Entry of command via menus or by just typing them in manually

We believe that all of these benefits help propel us towards a pedagogical CAS.

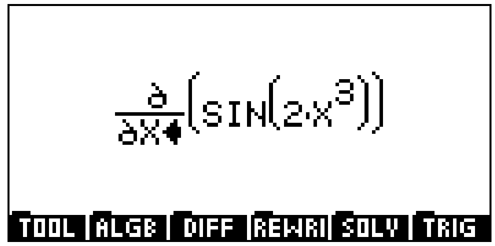

Figure 28

Figure 29

Figure 30

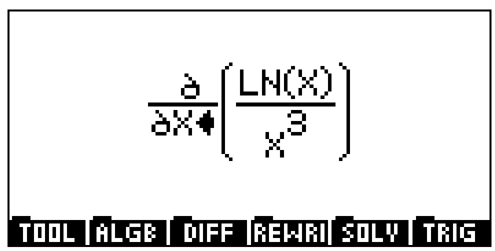

Figure 31

Figure 32