TECHNOLOGY TO ENHANCE PROOF IN LINEAR ALGEBRA:IS IT POSSIBLE?

Lisa Humphreys and Bin Wang
Mathematics and computer science department
Rhode Island College
Providence, RI 02908

Linear algebra is one of the most frequently taught mathematics courses in the undergraduate curriculum throughout universities and colleges in the United States. Normally it is taken after completing the calculus sequence. This gives the impression that one needs the knowledge of calculus to learn Linear Algebra. But in mathematics, it really should be the opposite: Calculus employs a lot of linear algebra, can't be self-contained and requires a vast knowledge of mathematics from other fields, while Linear Algebra is more self-contained and requires little from other fields.

So why do we require students to take calculus before the Linear Algebra? The reason Linear Algebra is placed after calculus in undergraduate curriculums is that there is a large leap in mathematical maturity needed to do a proof based course in Linear Algebra. Students see, usually for the first time, many proofs and abstract concepts. Students have little intuition about the concept of vector spaces for example. This is completely different from working solely in R, R^2 or even R^3 (these being the easiest and most traditional examples of vector spaces) as they did in calculus. Yet Linear algebra is placed before a course in abstract algebra because we do have the examples of R^n to pull from, although this usually provides little comfort to students who are grappling with more abstract vector spaces. In a much deeper sense, Linear Algebra requires a very different kind of thinking, in which mathematical structures became the key of the course, i.e. we are only interested in rules, not the objects that obey these rules. To think in terms of mathematical structures, which are represented by very dry rules or laws, is much more advanced than to think about specific objects. The first leap of thinking already happened in high school when they are asked to pass from arithmetic to high school algebra. Now they are going to be asked to do the same thing again: pass concrete linear relations to Linear Algebra, a special set of rules. When there is only structure left in the problems that has no concrete qualities, we need abstract thinking. Without training, we can only see the behaviors of the concrete objects, but not the rules the objects follow which usually are hidden and needed to be formulated.

This lack of intuition on the structures causes a mental blank. Because of the lack of intuition, students will have difficulty judging the correctness of the statements and choosing methods to proceed. This lack of intuition is actually a common difficulty for students taking all "abstract" algebras. Linear Algebra is just the first algebra they encounter. Thus to train the students to develop an intuitive sense and to avoid mental blankness is our main emphasis in linear algebra classes.

Computer software provides us one way to help develop intuition. In some sense, the computer's calculations provide another reality to students' mind, so that they can rely on. In linear algebra, students are presented with a mixture of calculations and proofs. In this presentation we will show how to use mathematical software such as Maple and Matlab to actually investigate and aid in developing intuitions that finally lead to the proofs. In this paper, we are going to present two different ways to use software to aid students in proof.

> *Utilizing the symbolic manipulation capabilities.*
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> Most software packages have very powerful algebraic capabilities. This can be used to perform unmanageable calculations and assist a student in completing a proof, especially using direct forms of proofs or "brute-force" methods. Theoretical results involving row operations and determinants will be examined.
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> *Investigations beyond the scope of most elementary textbooks.*
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> Software can also be used to investigate and help to discover results not typically found in an undergraduate linear algebra text book. These results, although not new to mathematicians, will introduce students to mathematical research at an elementary level. Determining if a conjecture is true or false is also an important building block of higher mathematics. Students must learn to understand and test hypotheses under different, non-trivial circumstances. We will examine how to utilize the computer in a search for counter examples, then to draw a logical conclusion.

We will now explore six different examples that can be used in a first course in linear algebra.

<u>Example 1</u>: Prove that the eigenvalues of the following matrix are 5 and 1.

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

Proof: To calculate the eigenvalues of the matrix A we look for λ such that $Av = \lambda v$. Thus we solve the following equation.

$$\det\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix} = 0$$

We can use the Maple command to verify that $\lambda = 5$ is a solution.

$$Determinant(\langle\langle 2-5, 2, 1\rangle|\langle 1, 3-5, 1\rangle|\langle 1, 2, 2-5\rangle\rangle);$$

$$0$$

Similarly we can verify that $\lambda = 1$ also works.

$$Determinant(\langle\langle 2-1, 2, 1\rangle|\langle 1, 3-1, 1\rangle|\langle 1, 2, 2-1\rangle\rangle);$$

$$0$$

Thus

$$\det\begin{pmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{pmatrix} = 0 \qquad \text{and} \qquad \det\begin{pmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{pmatrix} = 0$$

and we have shown that 5 and 1 are eigenvalues of matrix A.

Example 2: Prove that the following vectors

$$\vec{v_1} = \begin{pmatrix} \frac{2}{3-a} & 1 & 3 \end{pmatrix} \qquad \vec{v_2} = \begin{pmatrix} \frac{6}{4-a} & 1 & 4 \end{pmatrix} \qquad \vec{v_1} = \begin{pmatrix} \frac{6}{-1-a} & 1 & -1 \end{pmatrix}$$

are linearly independent for any for $a \neq -1, 1, 2, 3, 4$.

Proof: We would like to show that

$$\det\begin{pmatrix} \frac{2}{3-a} & 1 & 3 \\ \frac{6}{4-a} & 1 & 4 \\ \frac{6}{-1-a} & 1 & -1 \end{pmatrix} = \frac{20(a-1)(a-2)}{(3-a)(4-a)(-1-a)} \neq 0$$

for $a \neq 1,2,3,4,-1$. To proceed, we use Maple:

$$Determinant\left(\left\langle\left\langle\frac{2}{3-a}, \frac{6}{4-a}, \frac{6}{-1-a}\right\rangle \middle| \langle 1, 1, 1\rangle \middle| \langle 3, 4, -1\rangle\right\rangle\right);$$

$$-\frac{20\left(-3\,a+2+a^{2}\right)}{(-3+a)\,(-4+a)\,(a+1)}$$

$$factor(-3a+2+a^{2})$$

$$(a-1)\,(a-2)$$

This verifies the formula

$$\det\begin{pmatrix} \frac{2}{3-a} & 1 & 3 \\ \frac{6}{4-a} & 1 & 4 \\ \frac{6}{-1-a} & 1 & -1 \end{pmatrix} = \frac{20(a-1)(a-2)}{(3-a)(4-a)(-1-a)}$$

and completes the proof.

<u>Example 3</u>: Investigate the following, make a conjecture and then prove your conjecture.

$$\det\begin{pmatrix} a & 1 & 1 & \dots & 1 \\ 1 & a & 1 & \dots & 1 \\ 1 & 1 & a & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & 1 & a \end{pmatrix}_{n\times n}$$

Solution: We will begin with the 2 x 2 case and proceed trying to find patterns and connections. We will propose a general formula. Finally we will use mathematical induction to prove our result.

We start with a *2 x 2* matrix.

$$Determinant(\langle\langle a, 1\rangle|\langle 1, a\rangle\rangle);$$

$$a^{2}-1$$

$$factor\left(a^{2}-1\right)$$

$$(a-1)\,(a+1)$$

This yields:

$$\det\begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} = (a-1)(a+1)$$

Using the same Maple commands, we can continue to increase the dimensions of the matrices which yields to the following:

$$\det\begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix} = (a-1)^2(a+2)$$

$$\det\begin{pmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{pmatrix}_{4\times 4} = (a-1)^3(a+3)$$

$$\det\begin{pmatrix} a & 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 & 1 \\ 1 & 1 & a & 1 & 1 \\ 1 & 1 & 1 & a & 1 \\ 1 & 1 & 1 & 1 & a \end{pmatrix}_{5\times 5} = (a-1)^4(a+4)$$

Now we can make an educated guess to the general formula

$$\det\begin{pmatrix} a & 1 & 1 & \dots & 1 \\ 1 & a & 1 & \dots & 1 \\ 1 & 1 & a & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & 1 & a \end{pmatrix}_{n\times n} = (a-1)^{n-1}(a+n-1)$$

We will use mathematical induction to prove this formula.

The formula is true for n = 2. Suppose it is true for n = k. Then calculate $(k+1)\times(k+1)$ matrix.

Using the linearity of the determinant on the first column:

$$\det\begin{pmatrix} a & 1 & 1 & ... & 1 \\ 1 & a & 1 & ... & 1 \\ 1 & 1 & a & ... & 1 \\ ... & ... & ... & ... & ... \\ 1 & ... & ... & 1 & a \end{pmatrix}_{(k+1)\times(k+1)} = \det\begin{pmatrix} a-1 & 1 & 1 & ... & 1 \\ 0 & a & 1 & ... & 1 \\ 0 & 1 & a & ... & 1 \\ . & . & . & . & . \\ 0 & 1 & 1 & ... & a \end{pmatrix} + \det\begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & a & 1 & ... & 1 \\ 1 & 1 & a & ... & 1 \\ . & 1 & .. & a & 1 \\ 1 & 1 & 1 & ... & a \end{pmatrix}$$

Apply the row reduction to the second matrix to make the first column a pivot column:

$$\det\begin{pmatrix} a & 1 & 1 & ... & 1 \\ 1 & a & 1 & ... & 1 \\ 1 & 1 & a & ... & 1 \\ ... & ... & ... & ... & ... \\ 1 & ... & ... & 1 & a \end{pmatrix}_{(k+1)\times(k+1)} = \det\begin{pmatrix} a-1 & 1 & 1 & ... & 1 \\ 0 & a & 1 & ... & 1 \\ 0 & 1 & a & ... & 1 \\ . & . & . & . & . \\ 0 & 1 & 1 & ... & a \end{pmatrix} + \det\begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 0 & a-1 & 0 & ... & 0 \\ 0 & 0 & a-1 & ... & 0 \\ . & .. & .. & ... & 0 \\ 0 & 0 & 0 & ... & a-1 \end{pmatrix}$$

Therefore

$$\det\begin{pmatrix} a & 1 & 1 & ... & 1 \\ 1 & a & 1 & ... & 1 \\ 1 & 1 & a & ... & 1 \\ ... & ... & ... & ... & ... \\ 1 & ... & ... & 1 & a \end{pmatrix}_{(k+1)\times(k+1)} = (a-1)(a-1)^{k-1}(a+k-1) + (a-1)^k = \left((a-1)^k(a+k)\right)$$

This completes the mathematical induction.

<u>Example 4</u>: Show the matrix

$$A = \begin{pmatrix} b+4 & 1 & 1 \\ -4a & 1 & -a \\ -4b-12 & a-4 & -3 \end{pmatrix}$$

is not diagonalizable for any non zero a.

Proof: Use the Maple command to calculate its characteristic polynomial.

$B := \langle\langle b+4, -4a, -12-4b\rangle|\langle 1, 1, a-4\rangle|\langle 1, -a, -3\rangle\rangle;$
$\quad CharacteristicPolynomial(B, 'x')$

$$x^3 - (2+b)\,x^2 - \left(-2\,b - 1 - a^2\right)x - b - a^2\,b$$

$factor\left(x^3 - (2+b)x^2 - \left(-2\,b - 1 - a^2\right)x - b - a^2 b\right)$

$$-\left(x^2 - 2\,x + 1 + a^2\right)(-x + b)$$

This result shows that the characteristic polynomial is

$$C.P. = (b-x)(x^2 - 2x + 1 + a^2)$$

This polynomial does not have more than one real root, if $a \neq 0$, because the quadratic factor

$$x^2 - 2x + 1 + a^2 = 0$$

does not have a real solution for $a \neq 0$.

Example 5 Show the following 4 vectors

$$\vec{v_1} = \begin{pmatrix} 6 & (2-a) & 2(2-a) & 4(2-a) \end{pmatrix}$$
$$\vec{v_2} = \begin{pmatrix} 24 & (3-a) & 3(3-a) & 9(3-a) \end{pmatrix}$$
$$\vec{v_3} = \begin{pmatrix} 60 & (4-a) & 4(4-a) & 16(4-a) \end{pmatrix}$$
$$\vec{v_4} = \begin{pmatrix} 120 & (5-a) & 5(5-a) & 25(5-a) \end{pmatrix}$$

are linearly independent if and only if a is not 0, -1, or 1.

Proof: We need to show the determinant of the matrix with these 4 column vectors is non-zero.

This uses ``brute of force" calculation that usually is impossible by hands. Using Maple we can calculate

$Determinant(\langle\langle 6, 24, 60, 120\rangle | \langle 2-a, 3-a, 4-a, 5-a\rangle | \langle 2(2-a), 3(3-a), 4(4-a), 5(5-a)\rangle | \langle 4(2-a), 9(3-a), 16(4-a), 25(5-a)\rangle\rangle)$

$$-12\,a + 12\,a^3$$

$factor(-12\,a + 12\,a^3);$

$$12\,a\,(a-1)\,(a+1)$$

This shows that if

$$A = \left(\vec{v}_1 \vec{v}_2 \vec{v}_3 \vec{v}_4\right)^T$$

Then

$$\det(A) = 12a(a-1)(a+1)$$

Thus the determinant is non-zero for $a \neq 0,1,-1$. This completes the proof.

Example 6. Let $f(t)$ be a polynomial in t. Let

$$D_f = \det\begin{vmatrix} 1 & 1 & 1^2 & \dfrac{f(1)}{1-a} \\ 1 & 2 & 2^2 & \dfrac{f(2)}{2-a} \\ 1 & 3 & 3^2 & \dfrac{f(3)}{3-a} \\ 1 & 4 & 4^2 & \dfrac{f(4)}{4-a} \end{vmatrix}$$

Prove or disprove that $f(a) = 0$ if and only if $D_f = 0$.

Proof: The determinant of the following matrix

$$D_f = \det\begin{vmatrix} 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{vmatrix}$$

is called Vandermonde determinant. It is well-known that it is not zero because 1, 2,3, 4 are distinct numbers. But the determinant in question is not the Vandermonde determinant. So the solution could be different. If f

has degree less than 4 and divisible by $t-a$, $\dfrac{f(t)}{t-a}$ is a polynomial of degree 3 or less. It follows that the last column is a linear combination of previous column, then the determinant is zero. Thus we predict the counter example requires f has degree at least 4.

To verify our prediction, we use the maple software to calculate D_f for the f of degree 1, 2, 3, 4.

(a) For a linear polynomial $f(t)=t+b$,

$$Determinant\left(\left\langle\langle 1,1,1,1\rangle\middle|\langle 1,2,3,4\rangle\middle|\langle 1^2,2^2,3^2,4^2\rangle\middle|\left\langle\frac{(1+b)}{1-a},\frac{(2+b)}{2-a},\frac{(3+b)}{3-a},\frac{(4+b)}{4-a}\right\rangle\right\rangle\right);$$

$$-\frac{12\,(b+a)}{(-4+a)\,(-3+a)\,(-2+a)\,(-1+a)}$$

Thus

$$D_f=-\frac{12f(a)}{(a-1)(a-2)(a-3)(a-4)}.$$

The statement is true.

(b) For a quadratic polynomial $f(t)=t^2+bt+c$,

$$Determinant\left(\left\langle\langle 1,1,1,1\rangle\middle|\langle 1,2,3,4\rangle\middle|\langle 1^2,2^2,3^2,4^2\rangle\middle|\left\langle\frac{(1+b+c)}{1-a},\frac{(4+2\,b+c)}{2-a},\frac{(9+3\,b+c)}{3-a},\frac{(16+4\,b+c)}{4-a}\right\rangle\right\rangle\right);$$

$$-\frac{12\left(c+b\,a+a^2\right)}{(-4+a)\,(-3+a)\,(-2+a)\,(-1+a)}$$

Therefore

$$D_f=-\frac{12(a^2+ba+c)}{(a-1)(a-2)(a-3)(a-4)}=-\frac{12f(a)}{(a-1)(a-2)(a-2)(a-4)}$$

Thus $D_f=0$ if and only if $f(a)=0$. The statement is also true.

(c) For a cubic polynomial, $f(t) = t^3 + bt^2 + ct + d$

$$Determinant\left(\left\langle\langle 1, 1, 1, 1\rangle \middle| \langle 1, 2, 3, 4\rangle \middle| \langle 1^2, 2^2, 3^2, 4^2\rangle \middle| \left\langle \frac{(1 + b + c + d)}{1 - a}, \frac{(8 + 4b + 2c + d)}{2 - a}, \frac{(27 + 9b + 3c + d)}{3 - a}, \frac{(64 + 16b + 4c + d)}{4 - a} \right\rangle\right\rangle\right);$$

$$-\frac{12\left(d + c\,a + b\,a^2 + a^3\right)}{(-4 + a)\,(-3 + a)\,(-2 + a)\,(-1 + a)}$$

Hence

$$D_f = -\frac{a^3 + ba^2 + ca + d}{(a-1)(a-2)(a-3)(a-4)} = \frac{-12 f(a)}{(a-1)(a-2)(a-3)(a-4)}$$

The statement still holds.

(d) For a quartic polynomial $f(t) = t^4 + bt^3 + ct^2 + dt + e$

$$Determinant\left(\left\langle\langle 1, 1, 1, 1\rangle \middle| \langle 1, 2, 3, 4\rangle \middle| \langle 1^2, 2^2, 3^2, 4^2\rangle \middle| \left\langle \frac{(1 + b + c + d + e)}{1 - a}, \frac{(16 + 8b + 4c + 2d + e)}{2 - a}, \frac{(81 + 27b + 9c + 3d + e)}{3 - a}, \frac{(256 + 64b + 16c + 4d + e)}{4 - a} \right\rangle\right\rangle\right);$$

$$-\frac{12\left(-24 + 50\,a + e - 35\,a^2 + c\,a^2 + d\,a + 10\,a^3 + b\,a^3\right)}{(-4 + a)\,(-3 + a)\,(-2 + a)\,(-1 + a)}$$

Now we see the numerator is not $f(a)$.

The counter-example of the statement now can be

$$f(t) = t^4$$

In this case, $D_f = -12\dfrac{-24 + 50a - 35a^2 + 10a^3}{(a-1)(a-1)(a-3)(a-4)}$, but $f(a) = a^4$.

This shows D_f and $f(a)$ would not be zero simultaneously for some a.