

# HOW CAN I USE MAPLE TO HELP MY STUDENTS LEARN MULTIVARIATE CALCULUS?

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## Introduction

Maple is a powerful software program that can be used to help students learn math. With the advent of clickable-calculus, students are able to use Maple independently in a relatively short amount of time. In this article I will illustrate the kinds of things an instructor can do with Maple to help the students “learn math by seeing math”. Many of the illustrations are presented in the context of the adventures of Lisa the lovable lightning bug.

## Seeing Vector Decomposition

In the first lab, Lisa is in the  $xy$  – plane flying along the folium of Descartes given by  $2x^3 + 2y^3 = 9xy$ . In the lab, the students are guided through the commands required to plot Lisa’s path, the point  $(2,1)$ , and the unit tangent and unit normal vectors to Lisa’s path at the point  $(2,1)$ . All four of these objects have been plotted in Figure 1 below.

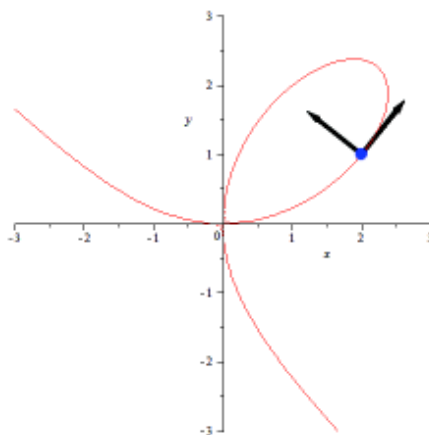


Figure 1: Lisa’s Path With Tangent and Normal Vectors

It should be noted that although Maple does much of the computational work, the students must still 1) know about implicit differentiation, 2) interpret the derivative as the slope of the tangent line, 3) know how to use the slope of a vector to find the components of the unit vector, and 4) know how to find a vector orthogonal to a given vector. So, although Maple helps a lot, the students still have to know some math to work through the process and ultimately pull it all together to get the plot above.

Enter Louis the liquidator – a very mean and nasty lightning bug! Louis, who spends a lot of time lurking in the shadows, waits for Lisa to pass by and then zaps her with his "force vector generation device". Louis is at the point  $R(-2,-1)$  and he zaps Lisa with a force of 3 Newtons when she is at the point  $P(2,1)$ . Lisa struggles against the force but is unable to resist it so she ends up at the point  $Q(5,0)$ . Given this information, students are required to use Maple (to the extent possible) to 1) find Louis' force vector,  $\mathbf{F}$ , 2) find Lisa's displacement vector,  $\overline{PQ}$ , 3) compute the work done by the force in moving Lisa from  $P$  to  $Q$ , and 4) compute the "effective force" and the "wasted force", i.e. the component of  $\mathbf{F}$  parallel to  $\overline{PQ}$  and the component of  $\mathbf{F}$  perpendicular to  $\overline{PQ}$ , respectively. Once they have found all these quantities, they then graph them (along with Lisa's path) on the same set of axes, thus seeing what vector decomposition actually "looks like". See Figure 2 below.

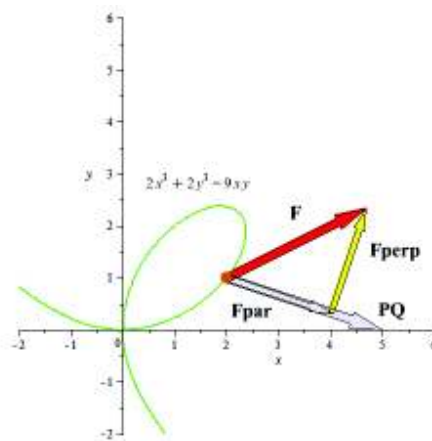


Figure 2: Louis' Force Vector (Decomposed) and Lisa's Displacement Vector

### Seeing a Trace

Maple is also quite useful in helping students visualize traces of surfaces. In the second lab, the students investigate the traces of several quadric surfaces using Cartesian, cylindrical, and spherical coordinates. For each surface, they plot the surface and several intersecting planes, each in turn. This is done in the context of two lightning bugs flying around in  $R^3$ , namely, Larry and Lisa. Larry is a *plane* old lightning bug, hence he always stays on a *plane*. Lisa, on the other hand, is much too sophisticated to stay on a plane. The lab starts with Lisa flying around on the hyperbolic paraboloid given by  $z = 2x^2 - y^2 + 1$  and Larry flying around on the  $xy$ -plane and desiring greatly to meet Lisa.

The students are asked to find the set of points in  $R^3$  that Larry should travel along in the hope of finding a *trace* of Lisa. This requires the students to visualize the hyperbolic paraboloid, the  $z = 0$  plane, and the curve of intersection of these two surfaces. Using Maple, the students can quickly plot both surfaces and rotate the graph to see the curve of intersection, which is the  $z = 0$  trace of the hyperbolic paraboloid. See Figure 3 below.

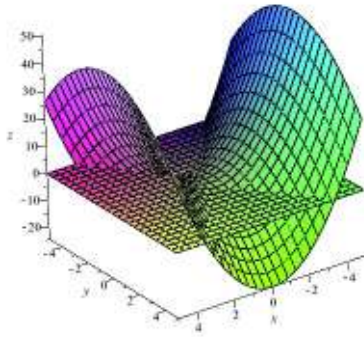


Figure 3: Lisa on Hyperbolic Paraboloid and Larry on  $z = 0$  Plane

Larry flies around all day long jumping from one coordinate plane to the other, but never sees a *trace* of Lisa. To make matters worse, Lisa decides to alter her flight pattern so that she will always be somewhere on the surface given by  $9x^2 + 4y^2 = 36$ . Students first convert this equation to its corresponding equation in cylindrical coordinates and then use Maple to plot it using polar coordinates. Upon converting to cylindrical coordinates the students find that Lisa's surface is given by  $r = \frac{6}{\sqrt{9\cos^2(\theta) + 4\sin^2(\theta)}}$ . They then plot

this surface and each coordinate plane in turn to find out where Larry must go to find a *trace* of Lisa. For each trace, the students include a graph of the surface and the plane, the equation of the trace, and the name of the trace (line, parabola, etc.) See Figure 4 below, which corresponds to the  $x = 0$  trace.

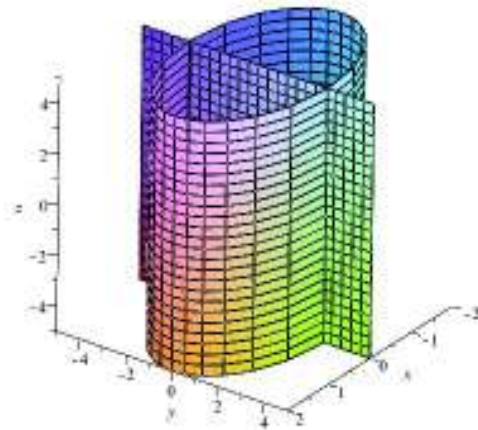


Figure 4: Lisa on Elliptic Cylinder and Larry on  $x = 0$  Plane

Foiled again! Larry flew around all day and did not see a trace of Lisa ... again! Lisa, being the adventurous sort, has decided to alter her flight pattern yet again. Now Lisa is

flying around  $R^3$  so she always stays on the surface  $-x^2 - y^2 + z^2 = 1$ . Larry is also getting more adventurous – or desperate – and decides to venture way out to the  $x = 1$  plane in a last ditch effort to cross paths with Lisa. With all this in mind, the students are asked to find the set of points Larry should visit on the  $x = 1$  plane in the hope of finding Lisa. Once again, the students must first use the coordinate transformation equations to convert the given equation to its corresponding equation in spherical coordinates. They do this “by hand” to find that Lisa’s surface is given by  $\rho = \pm \frac{1}{\sqrt{\cos^2(\varphi) - \sin^2(\varphi)}}$ . See

Figure 5 below for a graph of the  $x = 1$  plane and Lisa’s surface in spherical coordinates.

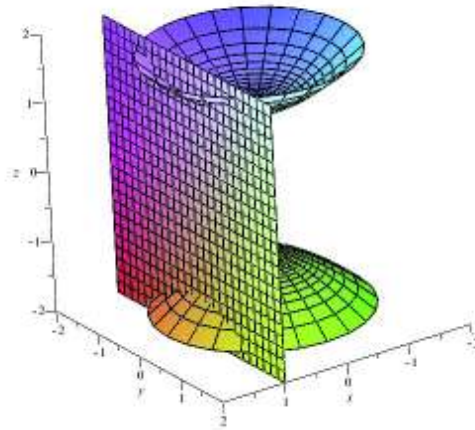


Figure 5: Lisa on Hyperboloid of 2-Sheets and Larry on  $x = 1$  Plane

Success at last! Larry the lonely lightning bug finally met Lisa the loveable lightning bug by going to the highest point below the  $xy$ -plane on the  $x = 1$  trace and hovering. When he saw the light (actually it was a flash) he traveled along the trace to investigate, and sure enough ... it was Lisa! What were the coordinates were Larry hovered? To see the  $x = 1$  trace and where Larry hovered, the students plot the  $x = 1$  trace and label the two axes appropriately. This is shown in Figure 6 below.

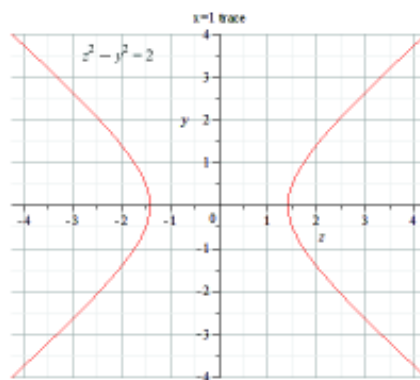


Figure 6: Lisa’s Trace on the  $x = 1$  Plane

### Seeing a Torus

In the third lab, students are given the parametric representation (in the parameters  $s$  and  $t$ ) of a specific torus and are required to determine the range of parameter values that will plot the torus exactly once, or that will plot only the bottom half of the torus, etc. Students typically do this by first thinking, and then plotting to either confirm or “refute and refine”. See Figures 7 and 8 below for examples of this.

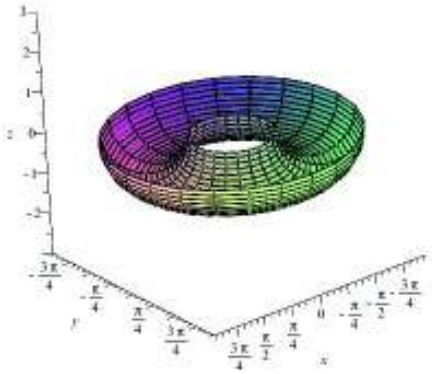


Figure 7: Bottom Half of Torus

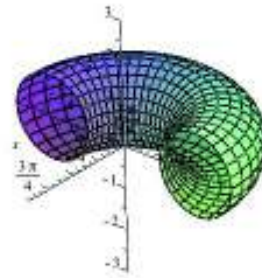


Figure 8: Part of Torus Behind  $yz$ -plane

Next, the students see what the family of  $s = \text{constant}$  and  $t = \text{constant}$  grid curves look like. They first plot each separately and then put the two families together to see the torus starting to take shape before their very eyes. See Figures 9 and 10 below.

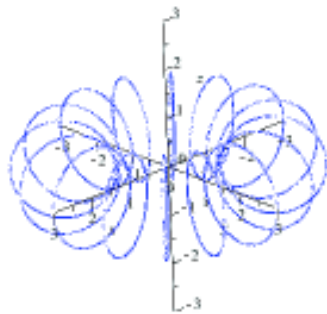


Figure 9: Grid Curves ( $s = \text{constant}$ )

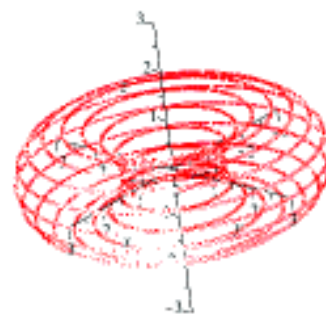


Figure 10: Grid Curves ( $t = \text{constant}$ )

When the two sets of grid curves are plotted on the same set of axes, the students begin to see how the torus can be plotted by first plotting grid curves and then splashing some

color on them to get a surface. See Figure 11 below where both sets of grid curves are plotted.

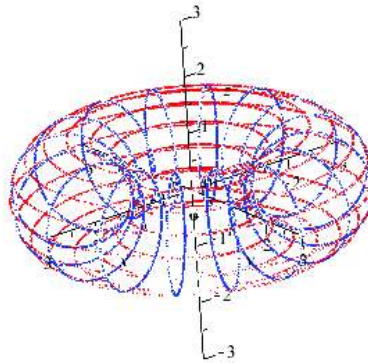


Figure 11: Both Sets of Grid Curves for the Torus

In the last section of this lab, students experiment with the more general expression for a torus given by  $\mathbf{r}(s,t) = \langle \cos(s) \cdot (a + b \cos(t)), \sin(s) \cdot (a + b \cos(t)), b \sin(t) \rangle$ . They plot this expression for various values of  $a$  and  $b$ . For each pair of values, they first plot the torus (playing around with the graph a bit in order to get a decent looking graph and viewing angle), and then name the graph what they think it looks like. (This is a “made up” name, not a technical name.) After plotting a torus for various values, the students must explain how changing  $a$  and  $b$  affects the shape of the torus. See figures below, which come from the above position vector with various choices of  $a$  and  $b$  as indicated.

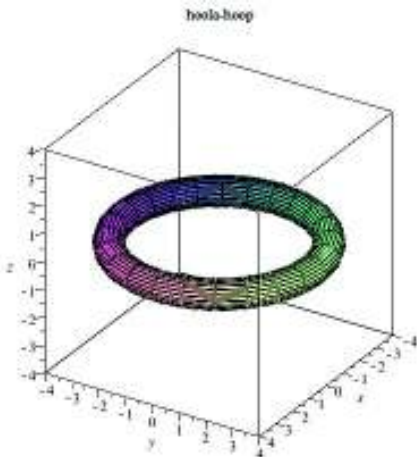


Figure 12: Torus,  $a = 3.5$  and  $b=0.5$

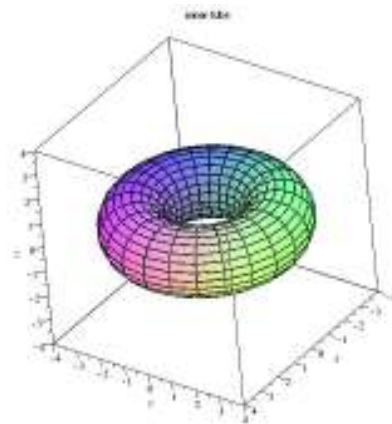


Figure 13: Torus,  $a = 2.5$  and  $b = 1.25$

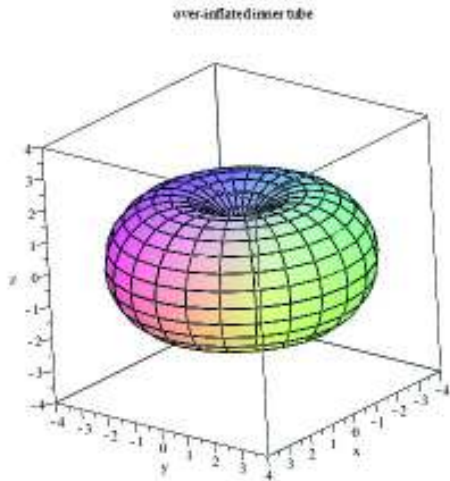


Figure 14: Torus with  $a = 2$  and  $b = 2$

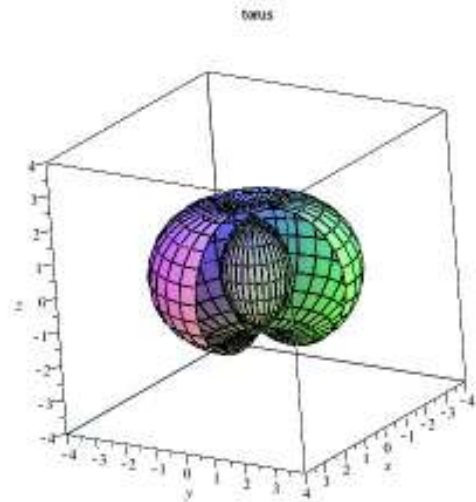


Figure 15: Torus with  $a = 1$  and  $b = 2$

### Seeing the Path of Steepest Descent

In this lab, the students explore some of the properties of the gradient of  $f$ ,  $\nabla f$ , the level curves of  $f$ ,  $f(x, y) = c$ , and the directional derivative of  $f$ ,  $D_u f(x, y)$ . They begin by

plotting the surface they'll be working with, namely,  $f := (x, y) \rightarrow 100 \cdot \left( 1 - e^{-\frac{x^2 + y^2}{2}} \right)$ .

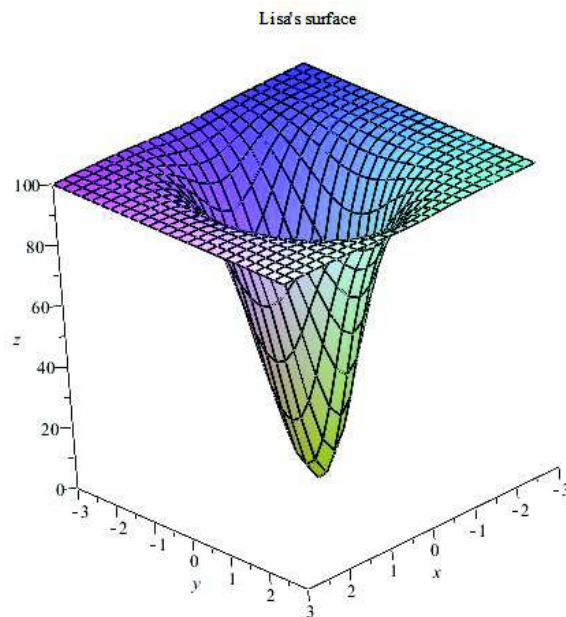


Figure 16: Lisa's Surface

The students then use Maple (to the extent possible) to help them a) plot the level curve of  $f$  through the point  $(1, -2)$ , find and plot the unit vector pointing in the opposite direction of the gradient at  $(1, -2)$ , and c) find and plot the beeline to the origin from the point  $(1, -2)$ . See figure below.

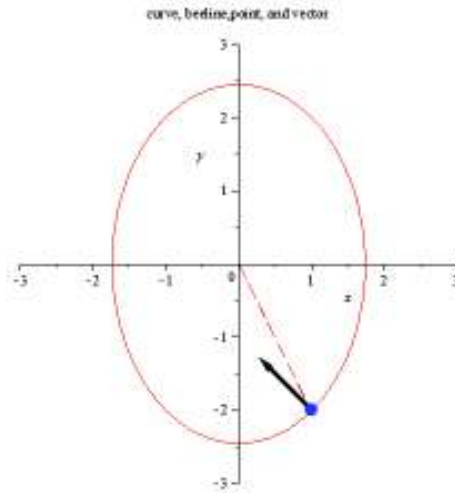


Figure 17: Level Curve, Unit Vector, and Beeline to Origin

Next, the students investigate which path gets Lisa to the origin (=the bottom of the surface) the fastest. They do this by using Maple to compute the directional derivative in the direction of the negative gradient vector and in the direction of the beeline to the origin. This reiterates the fact that Lisa will get to the bottom of the surface fastest if she always proceeds in the direction opposite the gradient vector. The students then use Maple as they work through finding the family of orthogonal trajectories to the level curves of  $f$  in order to find Lisa's path to the origin. See figure below.

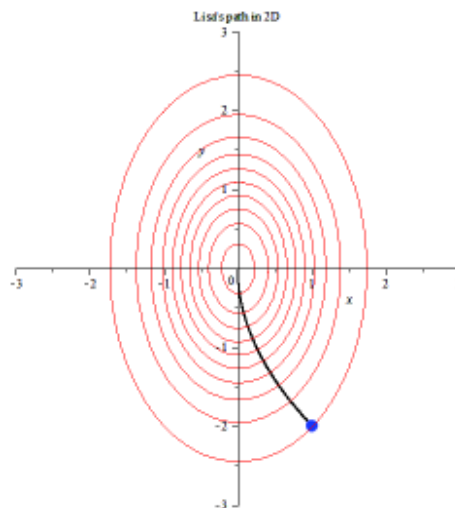


Figure 18: Lisa's Path in the  $xy$  - plane



And finally, the students use Maple (and their knowledge of space curves) to pull it all together and plot 1) the surface Lisa is sliding down, 2) Lisa's path in  $R^3$  as she slides down the surface, namely, the path of steepest descent, and 3) Lisa's path in  $R^2$ , namely, the projection of the path of steepest descent onto the  $xy$  – plane. See figure below.

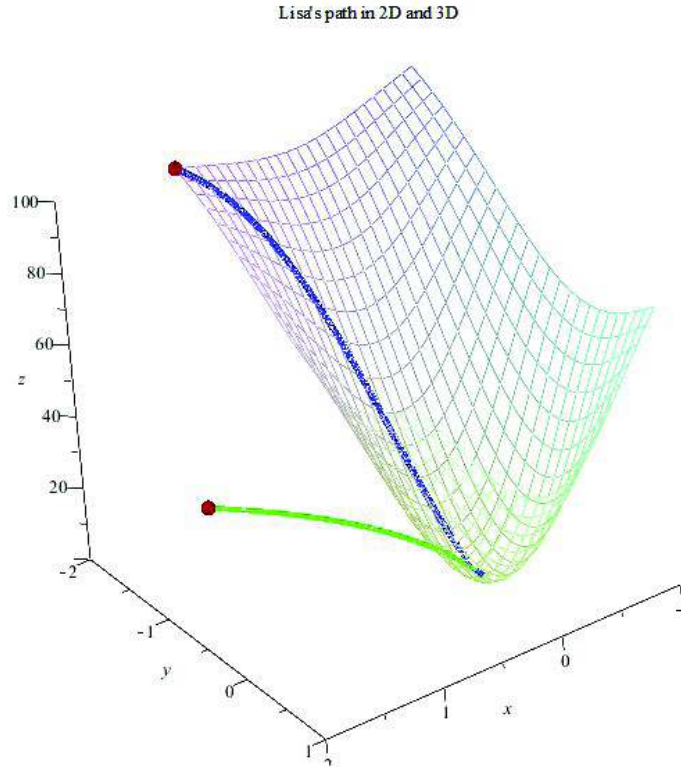


Figure 19: Lisa's Surface, Path of Steepest Descent, and Path in  $xy$  – plane

### **Conclusion**

Maple is a useful tool in multivariate calculus because it helps the students visualize the symbolic expressions they work with in class. It is also a versatile and powerful tool for plotting many different kinds of two-dimensional and three-dimensional objects (vectors, points, lines, planes, space curves, families of curves, and surfaces) on the same set of axes. By helping the students visualize the objects they are working with, Maple helps them learn the mathematics behind the symbols. Seeing is still believing!