USING PROJECTS TO DEVELOP CONCEPTS IN ALGEBRA/PRECALCULUS

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Abstract

Data-rich projects that require use of technology provide an effective way to develop students' understanding of different functions and mathematical techniques needed to solve real-world problems. In this paper, two project topics are presented: modeling height from bone length and predicting the population of an endangered species.

Introduction

In algebra and precalculus courses students learn about the behavior of a variety of functions. Textbooks often contain applications involving functions to drive home the point that these particular functions are useful in the real world, to showcase important features of these functions, and to provide an opportunity to apply algebraic techniques related to problem solving. However, rather than empowering students, applications are often seen as "just another word problem" or another mathematical hurdle for students to jump over.

Projects in which students have an opportunity to develop a model from data using the curve-fitting capabilities of graphing calculators or spreadsheets can provide students with a stronger connection to the functions they are studying and a deeper understanding of the properties of those functions. The discussion that follows illustrates how technology-reliant projects can enhance instruction in algebra and precalculus courses. During work on projects, students are forced to give up the role of passive learner. Instead they work collaboratively, grappling with mathematical ideas and problem solving. In addition, the real-world contexts of the projects give meaning to the symbols, graphs, and manipulations required to solve mathematical problems.

Linear Functions: Predicting Height From Bone Length

For example, contrast an exercise involving the formula $H = 2.38F + 61.41$ for predicting a person's height, H, in centimeters from the length of their femur, F, in centimeters to a project in which students take on the role of forensic scientists and determine a model for predicting height from data on height and femur length. Not only does the activity create more student interest, but students can go deeper into the topic of linear functions.

The premise for the project is that a hunter discovers partial skeletal remains that include femur and ulna bones. As a clue toward identifying the remains, students are asked to predict the person's height. But first, they need to create models that summarize the relationship between height, y, and bone length, x. Students are provided sample data from the Forensic Data Bank at the University of Tennessee on height (cm), femur length (mm) and ulna length (mm) of 29 female and 31 male skeletons. The scatterplot of height versus femur length shown in Figure 1 (data from [2, p. 90]) turns out to have linear form and hence a least-squares line can be fit to these data.

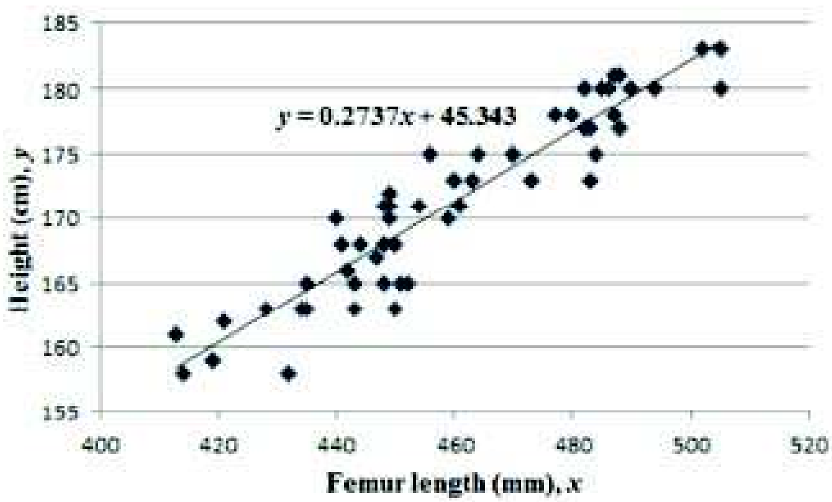

Figure 1. Fitting a line to height-versus-femur-length data.

Although students are readily able to identify the values of the slope and y-intercept of the model, interpreting these quantities in context often proves to be problematic. The y-intercept of this model indicates that a person whose femur length is 0 mm is predicted to be 45.3 cm tall, which makes perfect sense mathematically but is nonsense in the context of this problem. On the other hand, the slope does have meaning in this context: for each 1 mm increase in femur length, height is expected to increase by a 0.27 cm. Hence, for two people whose femurs differed by 3 mm, we would predict their heights to differ by 0.81 cm.

Next students separate the data by gender and fit separate models for males and females:

- Model for females: $y = 0.254x + 53.37$
- Model for males: $y = 0.234x + 65.05$

This provides an opportunity for students to compare the rate of change in height per unit increase in femur length (the slopes) for the men's and women's models. Figure 2 shows graphs of the two models on the same set of axes.

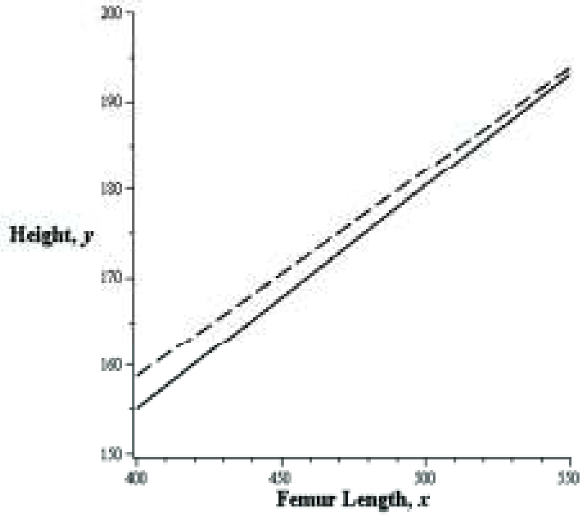

Figure 2. Graphs of models for males and females.

Asking students to identify which model, the men's or women's model, is represented by the dotted line forces them to grapple with graphical clues to selecting the model with the smaller slope. With some basic understanding of the models, students are now ready to answer questions such as those below:

1. Imagine that you are called in to advise law enforcement authorities on a case in which all that remains of a person is the femur bone. With nothing more to go on, you are unsure whether the person is male or female. If the length of this bone is 47 cm, how much difference would it make if you used the regression equation for predicting the height of a man and it turned out that the bone belonged to a woman? (Warning: Pay attention to the units.)
2. For what femur length would the regression equation for predicting a man's height give the same results as the regression equation for predicting a woman's height? (Show how to find the answer using algebra and then how you could check your result using graphs.) Is that result a reasonable femur length for an actual person? Explain.
3. If a man and a woman have femurs of the same length, which of them is likely to be taller? Justify your answer using graphs of the two regression lines, remembering to stay within reasonable bounds for femur lengths.
4. Returning to the mystery of the remains found by the hunter, assume that a skull found at the site indicates the deceased is male. Predict the person's height if the femur measures 474 mm.

These four questions are just a sample of the rich set of questions that can be generated from this context. Such questions push students to think more deeply about linear functions and the usefulness of such models in solving real-world problems.

Exponential Functions: Modeling the Kemp's Ridley Sea Turtle Population

The Kemp's Ridley Sea Turtle Population Project [1] consists of three activities. In the first activity students discover that data from an exponential function can be linearized by plotting log(y) versus x. Take, for example, $y = 4 \cdot 1.25^x$. A table of values and a plot of log(y) versus x appear in Figure 3.

x	y	log(y)
0	4.00	0.602
1	5.00	0.699
2	6.25	0.796
3	7.81	0.893
4	9.77	0.990
5	12.21	1.087
6	15.26	1.184

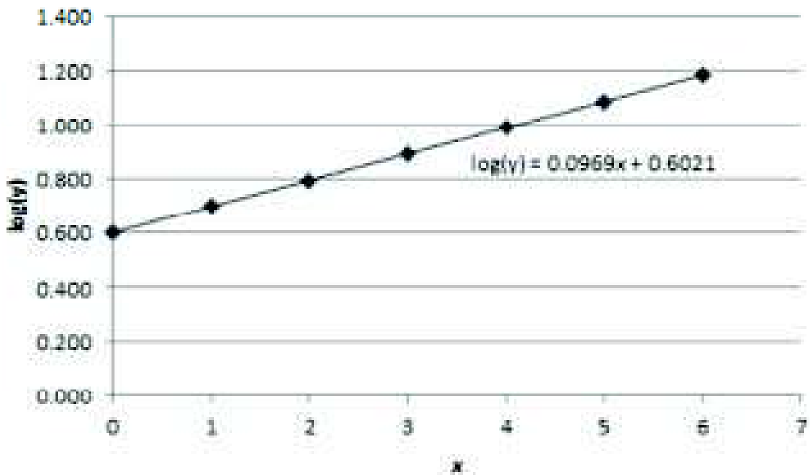

Figure 3. Table of values from an exponential function and a plot of log(y) versus x.

The exercise to work backwards and convert a function such as $\log(y) = 0.301 + 0.1461x$ into an explicit function of y in terms of x is challenging for students and requires that they apply several of the rules for exponents:

$$10^{\log(y)} = 10^{0.301+0.1461x}$$
$$y = 10^{0.301} \cdot 10^{0.1461x}$$
$$y = 2 \cdot (10^{0.1461})^x$$
$$y \approx 2 \cdot 1.4^x$$

Once students have learned how to apply a logarithmic transformation to linearize exponential data, they are ready to begin Activity 2, which deals with the decline of the Kemp's ridley sea turtle population.

The Kemp's ridley sea turtle population experienced a dramatic decline from 1947 to 1986 (see Table 1). It should be noted that it is impossible to estimate the size of the Kemp's ridley population directly. Young turtles and males stay at sea. Only adult females come ashore to lay their eggs. Hence, population estimates must be based on the nesting female population.

Year	Years since 1947, x	Nesting females, y	log(y)
1947	0	40,000 +	4.609
1968	21	5000	3.699
1970	23	2500	3.398
1974	27	1200	3.079
1986	39	621	2.793

Table 1. Data on the number of nesting females from 1947 – 1986 .

Given that a plot of log(y) versus x appears roughly linear, an exponential model is fit to the data. If the exponential model is fit using a TI-84 graphing calculator, then the result is $y = 37811 \cdot 0.895^x$, where y is the size of the nesting female population and x is the time in years since 1947. If Excel is used instead, then the result is $y = 37811 \cdot e^{-0.111x}$. This discrepancy in the form of the model provides an opportunity to discuss conversion from one form of exponential model to the other, a topic covered in most precalculus textbooks. The 10.5% annual decline can be easily obtained from the first form of the model. Either form can be used to predict how long it would take for the Kemp's ridley sea turtle to be, for all practical purposes, extinct.

In 1978, a binational (Mexican and U.S.) Kemp's Ridley Working Group was formed. The group's efforts resulted in increased protection at the main nesting beach at Rancho Nuevo in Mexico, promotion of a second nesting beach at Padre Island National Seashore in Texas, and an experimental "head start" program in which hatchlings were raised in captivity for their first year to decrease infant mortality. Dr. Rene Marquez, a member of the working group, developed the following model from data collected from 1978 to 1985 (see [1] for data):

$$\log\big(N(x)\big) = 2.89 - 0.0195x,$$

where $N(x)$ is the expected number of nesting females and x is the time in years since 1977. Students transform Dr. Marquez's model into the exponential function below

$$N(x) = 776.247 \cdot (0.956)^x,$$

and conclude that the Kemp's ridley population has continued to decline from 1978 – 1985 but at a slower annual rate of around 4.4%.

With the Kemp's ridley turtle population still in decline, environmentalists focused their investigations on the causes of sea turtle mortality. One major cause turned out to be drowning in the nets of shrimp trawlers. After December 1, 1994, all trawlers in U.S. waters from Virginia to Mexico were required to use turtle excluder devices (TEDs) year round. The effect of this change can be seen from the data in Table 2.

Year	1988	1989	1990	1991	1992	1993	1994	1995	1996
Number of nests	854	739	70	840	899	857	1153	1430	1288
Number of nesting females	342	296	312	336	360	343	461	572	515

Year	1997	1998	1999	2000	2001	2002	2003	2004	2006
Number of nests	1549	2413	2298	3778	3846	4194	5380	4463	7866
Number of nesting females	620	965	919	1511	1538	1678	2152	1785	3146

Table 2. Data on nesting females from 1988 – 2006.

Activity 3 begins with the analysis of the data in Table 2. Figure 4(a) shows a scatterplot of the log of the number of nesting females, $\log(y)$, versus the years since 1988, x, beginning with $x = 4$ (the year prior to the required use of TEDs). The equation of the regression line is $\log(y) = 0.072x + 2.209$. Transforming the linear model into an exponential model yields $y = 161.808 \cdot 1.180^x$. A graph of the exponential model superimposed on a scatterplot of *y* versus *x* is shown in Figure 4(b).

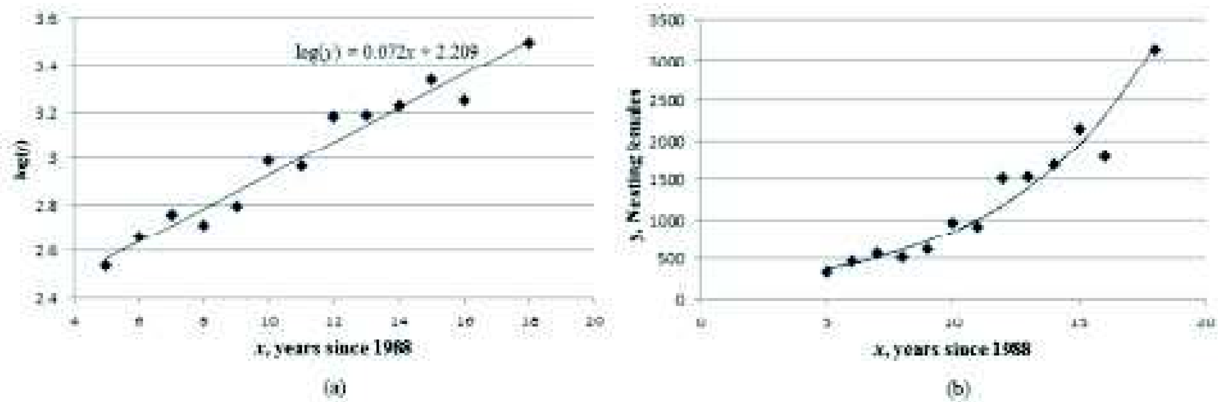

Figure 4. (a) Fitting a line to log(*y*) versus *x*; (b) Graph of *y* versus *x* and the exponential model.

These data indicate that the Kemp's ridley sea turtles appear to be rebounding. Based on the exponential model, the population is growing at an annual rate of 18%. There is one last question for students:

> Under the Endangered Species Act, the Kemp's ridley sea turtle will be upgraded from "endangered' to "threatened" once the population of nesting females at Rancho Nuevo reaches 10,000. On the basis of your model, predict when the Kemp's ridley sea turtle will be upgraded.

To complete the activity students must solve the equation $161.808 \cdot 1.180^x = 10{,}000$.

In this project, students have an opportunity to fit exponential models to data and to interpret the growth patterns as annual percentage rates of change. They find that applying a logarithmic transformation to an exponential model, $y = a \cdot b^x$, produces a linear model, $\log(y) = y' = \log(a) + \log(b) \cdot x$. In addition, they solve several equations involving exponential functions in order to make predictions about the decline or growth of the Kemp's ridley sea turtle population.

References

[1] Davis, Marsha, " Modeling the Kemp's Ridley Sea Turtle Population," Consortium Pull-Out Section 99, *The Newsletter of the Consortium for Mathematics and Its Applications*, Fall/Winter 2010.

[2] Moran, Judith, Davis, Marsha, & Murphy, Mary, *Precalculus: Concepts in Context*, 2nd Edition, Brookscole, (2004).