Examples, Counterexamples and Intuition

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Abstract

We consider how a WIKI can be used in upper-division online classes in order to help students critically evaluate whether a given statement is true or a given proof is correct.

Context

Online courses have become the medium of choice for students who cannot otherwise attend in a traditional classroom setting. Online distance education allows students to take courses in the convenience of their home or office, at their leisure, free of the distractions of campus life, without commute, while at the same time being provided with almost instantaneous access to the instructor and an abundance of online resources.

This paper considers a critical aspect of any upper-division mathematics class: Creating an intuition among students as to whether a given statement is true or false, or whether a proof is correct. The challenge herein of course is not unique to online classes, nor is it unique to any particular area of mathematics.

Students often find it quite difficult to produce a convincing proof for a given statement, much less to evaluate whether a sequence of statements that purport to be a proof actually is a proof. A significant amount of research has been devoted to teaching students how to prove theorems (e.g. Moore 1994). Much less is known about how students learn to distinguish a true statement from a false one, or a correct proof from an incorrect one. In this paper, we consider one method that can help students overcome this problem in online classes.

Proofs, Examples, Counterexamples and Intuition

We begin with considering two problems, one from Topology, and the other one from an Introductory Proof Technique class.

True or False (Baker 1997): *Every compact subset of a topological space is closed*? If true prove the statement, if false, provide a counterexample.

Correct or Incorrect (Smith et al. 2006):

Claim. For all $n \in \mathbf{N}$, $n^3 + 44n$ *is divisible by 3.*

"Proof." (i) $1^3 + 44(1) = 45$*, which is divisible by 3, so the statement is true for* $n = 1$.

(ii) Assume the statement is true for some $n \in \mathbf{N}$*. Then* $n^3 + 44n$ *is divisible by 3. Therefore* $(n+1)^3 + 44(n+1)$ *is divisible by 3. By the Principle of Mathematical Inudction, the statement is true for all* $n \in \mathbf{N}$.

The difficulty students encounter with this kind of problem is that the statement is not obviously true nor is it obviously false. In order to determine whether the statement about closedness is true or not requires intuition and a sufficiently large supply of examples. In a sense, this problem requires students to think outside of the box. The most obvious topological space – the real line with the usual topology – cannot provide a counterexample, as all compact subset of the real numbers are closed and bounded. The fact that the statement is true for subsets of the most commonly studied topological space does not mean that the statement is true for all topological spaces. In the problem above, a suitably chosen T_1 space would present a counterexample that disproves that statement.

In the example of the claim and the proof, the proof very nicely follows the pattern for a proof by induction, yet it fails to explain how the statement for n+1 follows from the statement for n. Here, students are often mislead by the apparent flawless look of the proof.

In answering those questions, the student experiences both the scope and the limitations of Theorems. For example, students would have seen the proof of the theorem "*In a Hausdorff topological space, every compact set is closed.*" A properly chosen counterexample shows that the hypothesis that the space is a Hausdorff space cannot be removed.

For a mathematician who is familiar with the subject matter, the problems provide no challenge: We can fairly easily determine whether the proof at this level of mathematics is valid, or a given statement is correct. From the student's perspective, the decision is not so evident. In this respect, we are asking the student to take on a new role in the quest for knowledge. We are asking the student to transition from a passive consumer of the pre-fabricated algorithms and procedures of Calculus and other lower-division courses, to being a creator of knowledge.

Of course, the answer to any of the true-false questions one would expect in an undergraduate text is well-known. The student can expect that he or she will eventually find the right answer. In this respect, the student who is faced with problems like the ones above does not "create" knowledge for the mathematics community at large. In this context, the student creates knowledge on a small scale for himself or herself or for the class community. In the process of doing so, the student must draw on intuition, experience, examples and on the knowledge of classmates.

The research mathematician finds himself or herself in a different situation: Here, it is not always clear whether an answer can be found at all. However, research indicates that even research mathematicians rely heavily on examples and on intuition when deciding whether to attempt to prove or disprove a statement (Fischbein 1982), and we all value the friendly critique of our colleagues in seminars and at conferences.

While the majority of mathematics students will never do research in mathematics, the value of exercises like the two described in the introduction is that it allows students to experience knowledge and learning from a different perspective, namely from the perspective of a creator of knowledge.

The quest for truth… What role do students play in generating knowledge?

In examining the epistemic beliefs of young adults, Baxter-Magolda's (2001) conducted a long-term study of college-aged students through adulthood. She found that entering college students typically see themselves as recipients of knowledge. A typical expectation is characterized by the quote: "I'm here to learn and not to think" (Baxter-Magolda 2001). Later, these young adults change their epistemic belief: They understand that they do have a role in creating knowledge, rather than simply consuming knowledge. Baxter-Magolda calls the transition period from the earlier stage to the later one the *crossroads*.

We observe the similarity in students' beliefs about their learning as they progress through the mathematics curriculum. In lower-division courses (Calculus, Linear Algebra, Differential Equations), students expect to be provided with clear rules that tell them what to do. The purpose of most upper-division is to provide students with the opportunity to prove theorems. Any faculty member who has ever taught these upper-division courses can attest to the difficulty that students face when first confronted with need to prove a theorem. The typical lament "I can do math, but can't do proofs" clearly mirrors Baxter-Magolda's learning-versus-thinking quote. We therefore expect that mathematics students make a similar transition in their epistemic beliefs about mathematics, as the subjects of Baxter-Magolda's study make in their transition from teenager to adulthood.

Providing students with the opportunity to conjecture whether a statement is true or false (and then justify their reasoning), or providing them with a sequence of statements that purport to be a proof (and making students decide whether the proof is valid) therefore creates situation where students are put in the role of creators of knowledge. We will next investigate one possible method by which this can be accomplished in an online class.

Learning as a social process

It has long been observed that learning is a social process; the learning of mathematics is no exception (Thurston 1994). The literature about online class design consistently notes that creating a community of learners in online classes is key to success and student learning. An upper-division mathematics class has therefore the same requirements for online community as any other online class has. The necessity to create classroom community in online course has been discussed at length in the literature (Pomper 2007).

The purpose of this discussion is to show how the resources of online classes can be used to guide students through the crossroads process. Because this transition is a learning process, we cannot expect that a student makes this transition alone. A well-designed course at this level should provide mechanisms in which students can exchange ideas and learn with and from one another. In the next section, we will describe how one such feature can be used to guide students through the crossroads.

What is a wiki and how is it used in this context?

A wiki is a website that allows fast and collaborative editing of its contents. Many course management systems include a wiki feature of some sort. In the context of an online topology class, the instructor assigned approximately ten true-false (and provide proof or counterexample) questions per week. In the context of an online Introductory Proof Methods (Bridge) course, between three and five claim-proof-pairs were assigned in any given week of the semester. The instructor provided specific guidance of what the end-result should look like:

Each assigned statement should be transferred from the textbook into the wiki, students should discuss whether the statement is true or false, and should arrive at a consensus. Then they should either create a proof of the statement (if the statement is true) or provide a counterexample (if they decided that the statement is false). In examples where students were asked to evaluate the validity of a proof, a corrected or improved version of the proof was also requested.

Provided below are some examples of how students chose to discuss the proposed statements or proofs. One should note that students' responses have been significantly modified for the purpose of replicating the gist of the discussion in a manageable format.

Claim. For all $n \in \mathbf{N}$, $n^3 + 44n$ *is divisible by 3.*
"Proof." (i) $1^3 + 44(1) = 45$, *which is divisible by 3, so the statement is true for* $n = 1$.
(ii) Assume the statement is true for some $n \in \mathbf{N}$. *Then* $n^3 + 44n$ *is divisible by 3. Therefore* $(n+1)^3 + 44(n+1)$ *is divisible by 3. By the PMI, the statement is true for all* $n \in \mathbf{N}$.

Student 1: This proof has some valid points, but the writer failed to define S, and made other mistakes as well. Let S = { $n \in N$| P(n) is true for all natural numbers n}. The first part of step (i) is fine, but the writer didn't finish by stating that $1 \in S$. Then, in step (ii), the writer should not have stated "Assume," but should have stated "Suppose....for all n

∈S," instead of "....for some n∈N" and the statements that follow should be "If.... Then...." Step (iii) needs to be finished with the conclusion that S = N.

Student 2: I think this proof is a good proof, but I agree with Student 1, this proof needs to be presented in the proper format.

Student 3: I don't think the specifications are the problem. They can be filled in by the reader somewhat. The problem here is that we're not shown any reason for the n+1 expression to be true. It just goes straight from this is true for n and then it says this is true for n+1 without giving any reasoning. It's not complete, and it's not a good proof.

Student 2: I still like this proof and think it should get an A, or at least a B+.

Student 3: I disagree. I think the proof needs another step showing why (n+1)^3 + 44(n+1) is true given than n^3 + 44n. They're just flinging out a random statement, and it's just an outline, not a proof. There's no real logic presented here. I'd give it at most a C.

The discussion illustrates that some students first focus on the format of the proof, and fail to consider the content. Students 1 and 2 almost exclusively focus on implicit assumptions that are not stated and fail to see that actually nothing was proved. As discussion progresses, Student 3 notes the gap in the reasoning. In further discussion, students come to an agreement that the proof has no merit.

Likewise in the context of the online topology class, the correct answer was only obtained through discussion. Again, students' responses are heavily redacted to highlight the gist of the discussion.

True or False: *Every compact subset of a topological space is closed*?

Student 1: This is true. We proved that the closed intervals [a, b] are compact.

Student 2: But this does not mean that every compact set is closed. Shouldn't we use the Heine-Borel Theorem?

Student 3. I agree with Student 1. The Heine-Borel Theorem says that a set is compact if and only if it s closed and bounded. So if a set is compact, it must be closed.

Student 1: Isn't that what I said?

Student 5: Heine-Borel Theorem only talks about the space (R, U). If we look at a real easy space, like X ={a, b, c} with topology T = { ∅, X, {a} }, then {a, b} is compact and it's not closed.

Student 1: OIC. We have to look at all the possibilities, not just the usual.

The examples show that wikis can be used to engage students in critically evaluating the correctness of statements. More importantly, throughout the semester, we see that a gradual adjustment of student attitudes towards learning. At the beginning of the semester, some students attempt to answer these types of questions either by only providing a yes/no answer, or by citing a statement that can be found in the book, regardless whether this answer actually provides a counterexample, or determines whether a proof is indeed correct. For example, in evaluating a conditional statement of the form $P \to Q$, students often refer to Theorems that assert the converse as evidence that $P \to Q$ is false. It is also interesting to note that at the beginning of the semester, students state their opinion and wait for the instructor to evaluate which one is correct. Later in the semester, students are much more likely to critique each other's statements, and to adjust their reasoning based on this critique.

Future Work

While at the end of the semester all students are typically engaged in critically evaluating the statements, it is not clear whether there is any lasting effect on the students' epistemic beliefs about their role in mathematics. Will these students return to their belief that their role is simply to absorb the knowledge that is presented, or will they continue to be critical observers who have a role in shaping knowledge?

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