The use of Maple and Calculus Concepts to Solve an Optimization Problem in a Physical Phenomenon.

Somasundaram Velummylum and Jessica Mong
Claflin University
Department of Mathematics and Computer Science
400 Magnolia St
Orangeburg, SC29115
svelummylum@claflin.edu
jmong@claflin.edu

Abstract:
The Rainbow is one natural phenomenon whose formation depends on factors including angles of deflection, incidence and refraction. *The* Derivative Test, *Properties of light,* and *Maple* will be applied to derive the minimal angle of deflection which is an important characteristic in visualizing the rainbow.

Discussion:

We expect that the audience understands the basic physical terms:

1. Reflection
2. Refraction and
3. Deflection or Deviation of Light rays.

The rainbow is a very fascinating natural occurrence. In this discussion, we will attempt to familiarize ourselves to some extent about how the rainbow is formed. Certain conditions need to be satisfied for the rainbow to be formed. We will discuss two conditions in this paper.

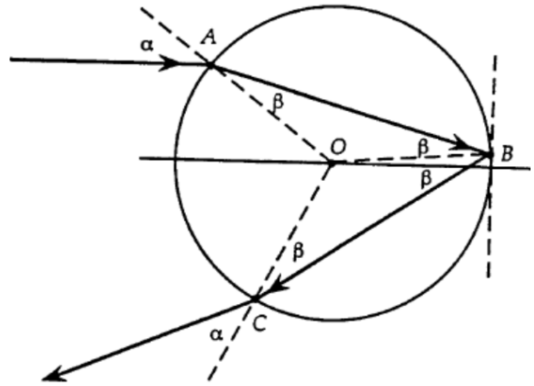

Figure1: Rain Drop

<u>*Refer to diagram above*</u>. Rainbows are formed when raindrops in the air both reflect and refract light rays from the sun. Let us follow the reflections and refractions caused by a raindrop.

1. The ray of light travelling through air hits the surface of the raindrop at point A and is refracted. This ray is bent towards the normal at A.
2. The refracted ray then travels through the raindrop and hits the back of the rainbow at point B and is reflected. The ray of light is bent away from the normal.
3. The reflected ray travels through the raindrop again and hits the raindrop at point C. The ray is then refracted back to air, and is bent away from the normal.

Deflection is the change in direction of light rays. Deflection therefore occurred at points A, B and C in the raindrop. Applying geometry, deflection at:

a. Point A $= \alpha - \beta$
b. Point B $= 180 - 2(\beta)$
c. Point C $= \alpha - \beta$

Looking at the figure again, we see that the ray of light enters the raindrop at point A and leaves it at point C. If we extend the rays of light at point A and at point C using imaginary lines, we see that they meet at the back of the raindrop. As a result we can calculate the total angle of deflection which is a sum of deflections at points A, B and C.

$$= (\alpha - \beta) + (180 - 2\beta) + (\alpha - \beta)$$

$$= 180 + 2\alpha - 4\beta$$

The ray exiting at point C is the ray of light that the observer sees. However, it can only be seen as the rainbow if the ray's total deflection is equal to the ***angle of minimum deflection***.

We will apply the concepts of Maximization and Minimization from Calculus.

Primary equation , where α , β are angles of incidence, refraction respectively as shown in the above diagram.

Secondary Equation $\longrightarrow$ Law of Refraction: $\frac{\sin(⍰)}{\sin(\beta)} = k$, where k is a constant.

We will use the Maple Software to do all calculations.

Key to the Maple commands used:

diff (d(⍰),⍰);	The '**diff**' command differentiates the d function with respect to ⍰
solve (equation, variable);	The '**solve**' command solves an equation for its unknown variables.
convert (expr, radians, degrees);	The '**convert**' command converts an expression from its original form (radians) to a new form (degrees)
evalf (expr);	The '**evalf**' command evaluates expressions containing constants such as π and *e*
subs (⍰ = a, expr);	The '**subs**' command substitutes a for ⍰ in the expression expr

Remember: d(α) = 180 + 2(α) - 4(β)

Since Maple operates automatically using radians instead of degrees, the deflection function becomes:

d(α) = π + 2(α) - 4(β)

d *is a function of both* (α) *and* (β).
However, from the <u>*Law of Refraction :*</u>

we know that (β) *can be expressed in terms of* (α)
We will need to keep this fact
in mind whilst doing calculations.

<u>***First***</u>, *we find the derivative of* ***d***(α).

\>
\> *with*(*Student*:-*Calculus1*); *infolevel*[*Student*[*Calculus1*]] := 1

[*AntiderivativePlot*, *AntiderivativeTutor*, *ApproximateInt*, *ApproximateIntTutor*, *ArcLength*, *ArcLengthTutor*, *Asymptotes*, *Clear*, *CriticalPoints*, *CurveAnalysisTutor*, *DerivativePlot*, *DerivativeTutor*, *DiffTutor*, *ExtremePoints*, *FunctionAverage*, *FunctionAverageTutor*, *FunctionChart*, *FunctionPlot*, *GetMessage*, *GetNumProblems*, *GetProblem*, *Hint*, *InflectionPoints*, *IntTutor*, *Integrand*, *InversePlot*, *InverseTutor*, *LimitTutor*, *MeanValueTheorem*, *MeanValueTheoremTutor*, *NewtonQuotient*, *NewtonsMethod*, *NewtonsMethodTutor*, *PointInterpolation*, *RiemannSum*, *RollesTheorem*, *Roots*, *Rule*, *Show*, *ShowIncomplete*, *ShowSolution*, *ShowSteps*, *Summand*, *SurfaceOfRevolution*, *SurfaceOfRevolutionTutor*, *Tangent*, *TangentSecantTutor*, *TangentTutor*, *TaylorApproximation*, *TaylorApproximationTutor*, *Understand*, *Undo*, *VolumeOfRevolution*, *VolumeOfRevolutionTutor*, *WhatProblem*]

$$infolevel_{Student:-Calculus1} := 1$$

\> $d := \alpha \rightarrow \mathrm{Pi} + 2\cdot\alpha - 4\cdot\left(\arcsin\left(\frac{1}{1.33}\cdot\sin(\alpha)\right)\right);$

$$d := \alpha \rightarrow \pi + 2\,\alpha - 4\arcsin\left(\frac{\sin(\alpha)}{1.33}\right)$$

\> $diff(d(\alpha), \alpha);$

$$2 - \frac{3.007518797\cos(\alpha)}{\sqrt{1 - 0.5653230821\sin(\alpha)^2}}$$

\> $solve(\mathbf{(1.3)} = 0, \alpha);$

$$-1.039952985, 1.03995298$$

\>

Because Maple operates automatically using radians instead of degrees, I would have to convert occasionally from radians to degrees and vice versa.

> $evalf(convert(\{\mathbf{(1.4)}\}, degrees));$

$$57.29577950\{-1.039952985, 1.039952985\}\ degrees$$

>

There are two critical points obtained: (α) = -59.6 and 59.6.

Next, we will use the 1st derivative test to determine which value of (α) gives the **Angle of Minimum Deflection**.

I will be using -60 degrees, 0 degrees and 60 degrees (converted to radians) as sample points in my first derivative test.

> $convert(60\ degrees, radians);$

$$\frac{1}{3}\pi$$

> $evalf\left(subs\left(\alpha = -\frac{\pi}{3}, \mathbf{(1.3)}\right)\right);$

$$0.01863624$$

> $evalf(subs(\alpha = 0, \mathbf{(1.3)}));$

$$-1.00751879$$

> $evalf\left(subs\left(\alpha = \frac{\pi}{3}, \mathbf{(1.3)}\right)\right);$

$$0.01863624$$

Interval	***Sample Point***	***Sign of*** $\mathbf{D}'(\alpha)$	***Behavior***
$-\infty < \alpha < -59.6$	-60	$+\ ve$	*Increasing*
$-59.6 < \alpha < 59.6$	0	$-ve$	*Decreasing*
$59.6 < \alpha < \infty$	60	$+\ ve$	*Increasing*

>

From the 1st Derivative Test Table above, it can be seen that (α) = -59.6 degrees gives maximum deflection.

The test therefore confirms that (α) = **<u>59.6</u>** <u>degrees</u> is the angle of incidence that produces the **Angle of Minimum Deflection**.

Now, let us find out the value of the **Angle of Minimum Deflection**.

> $evalf(convert(59.6\,degrees, radians));$

$$1.04021623$$

>

We will determine the **Angle of Minimum Deflection** by substituting (α) = 1.040216234 into the d(α) function.

> $evalf(d(\mathbf{(1.11)}));$

$$2.39954189$$

> $evalf(convert(2.399541896\,degrees));$

$$137.4836234\,degrees$$

We have been able to derive the **Angle of Minimum Deflection** as **<u>137.48</u>** <u>degrees</u>

Let's reinforce our findings with the graph of d(α) against (α)

> $plot(d(\alpha), \alpha = 0.5\,..3, d = 2.35\,..6);$

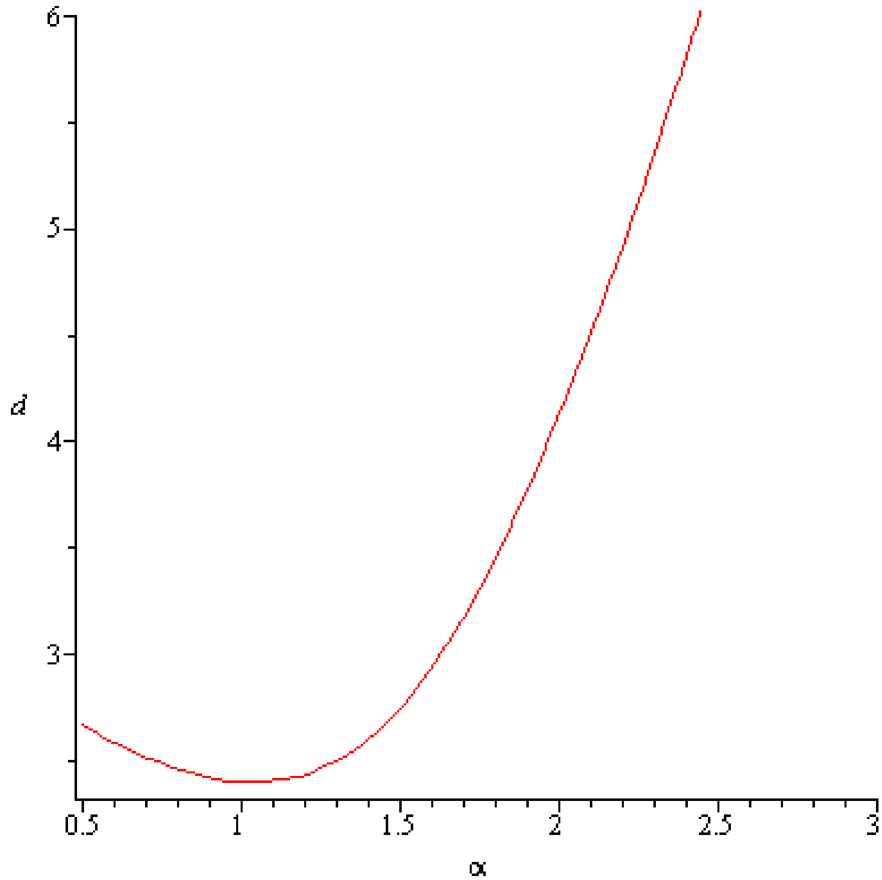

Figure2: Deflection Plot

Conclusion:

Calculus and Maple software can be used to solve real world problems such as Optimization problems in Physical Sciences and other fields. The usefulness of the Maple Software in Calculus and Mathematics in general, cannot be ignored.

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