TEACHING PARTIAL DIFFERENTIAL EQUATIONS USING TECHNOLOGY

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Abstract: We give a description of an advanced mathematics course at the University of Houston – Downtown where students are given an opportunity to generate and discover rather than to passively receive knowledge. The course material is motivated through projects that involve "real world" applications where students use technology to solve various problems and present their results.

1. Introduction.

In the traditional approach to teaching advanced mathematics courses, most of the class time is spent in one-way communication with an instructor lecturing and the students watching, listening, taking notes and working individually on homework and in-class assignments. The main focus of this paper is to give an example of an advanced course at the University of Houston – Downtown (UHD) which has provided an environment in which students have been given an opportunity to take an active role in the course. The students are encouraged to generate and discover rather than to passively receive knowledge. In particular, the course involves active and cooperative learning techniques in which students work in small teams on solving various research problems, they discuss, debate, reflect, learn by teaching each other, use technology and present their results. We believe that such instructional techniques promote interest in the subject and that students learn more material, retain information longer and, overall, enjoy the class more.

The main idea of this project-oriented course is on motivation of the material traditionally studied in the course on differential equations. Differential equations play a prominent role in physics, engineering, chemistry, biology, economics and other disciplines, and hold an important place in both pure and applied mathematics.

During the fall semester of 2008, I wrote a proposal for a grant at UHD (QEP – Quality Enhancement Plan) with an idea of introducing projects in our junior course on ordinary differential equations (Math 3301). I taught this one-semester course on several occasions earlier with the main focus of our syllabus on mastering techniques for explicit solving. At that time this was the only course on differential equations that our department was offering and the main motivation of my proposal was to give students a chance to learn that differential equations explain various physical phenomena and that their importance is in the applications which they describe. The QEP proposal was funded and I implemented it in spring and fall semesters of 2009 in Math 3301. In fall of 2009, our students were given a chance to attend a special topic course on Partial Differential

Equations. The course became very popular among our students and is now a regular elective course (Math 4304). During the fall semester of 2009, my colleague, Dr. Jeong-Mi Yoon suggested that we expand my QEP proposal and introduce projects within our two courses on differential equations – she developed projects that were implemented in Math 3301 and I wrote projects that were implemented in Math 4304. The proposal was funded and we implemented it during the spring semester of 2011.

The main focus of projects in Math 4304 is on understanding of various examples of partial differential equations with specific applications in gas and fluid dynamics. Since the employment market in greater Houston area seeks people with skills in these areas, it is important for our students to become familiar with mathematical methods for analyzing these problems. Such research experience is also valuable and beneficial to students who wish to pursue graduate degrees, as the nearby universities with graduate programs (Rice University, University of Houston, Texas A & M University, University of Texas at Austin, Southern Methodist University) offer a variety of advance studies in computational and applied mathematics.

2. Project Implementation in Math 4304.

Teaching differential equations, at the undergraduate level, usually assumes teaching techniques for their explicit solving. However, differential equations that model ``real world problems" are often too complicated and these standard techniques are usually not applicable. Consequently, one has to develop computational methods for approximate solving of these equations. Unfortunately, instructors often do not have enough time to cover such topics within the course and through the above mentioned projects our students are now able to actively and effectively learn in a short time about computational techniques for approximate solving of differential equations.

Three major components of each proposed project are

- derivation of partial differential equations which describe particular physical phenomena: these equations are simplified enough so that their basic properties could be also understood and analyzed theoretically,
- study of numerical methods for approximate solving: students derive numerical (finite difference and finite volume) methods and implement their own codes using Matlab, Maple or Mathematica; prior knowledge of these computer algebra systems is not required, as the codes involve only basic mathematical and logical operations, which students learn by ``doing the thing", and
- writing a complete report using MS Word, Latex or any other software.

We believe that nature of the real world examples highly motivates our students and enhances course delivery by engaging students in active and cooperative learning. After all, teaching should not be directive and based on a one-way communication, but developmental, problem-driven and student-oriented.

Duration of the whole project is around three weeks. During the first week, the students are given a detailed manual, written by the instructor, which describes:

- differential equations which model various physical examples, derivation of these equations using physical principles, some features of their solutions that could be understood analytically,
- formulations of basic numerical methods which are used for approximate solving of these equations (finite difference and finite volume), and
- computer code, written in Matlab, Maple or Mathematica, which is used for approximate solving.

There are five projects developed for Math 4304:

- aerospace engineering,
- secondary oil recovery,
- traffic flow,
- wave equation, and
- heat equation.

After the one-week introduction, the students select projects depending on their interests and they work in small teams (of three or four) for the remaining two weeks. This aspect of the project builds confidence in students' ability to do and apply mathematics. Unlike the usual methods of course delivery where students are passive recipients of information, the research element of our project enhances course delivery through active learning strategies such as development of communication and analytical skills, skills of working with others, skills of using technology, and promotes collaborative learning and peer evaluation. During this period, the students work on their research problems outside of the class time. These activities create an opportunity for students to reach their full potential, achieve educational excellence and prepare for transition from the classroom to the work/graduate school environments. At the completion of the course, each team is expected to write a report and present the summary of their project in the form of a 15-20 minute power-point presentation.

The material that the instructor offers for each project consists of a complete set of notes from selected textbook chapters on differential equations (by Evans [2], Haberman [4] and Kevorkian [9]) and numerical methods (by LeVeque [10, 11]), relevant research articles (by Jegdic [5, 6, 8] and Jegdic & Jerrard [7]) and information about software packages that are used for numerical simulations (by Davis [1], Garvan [3] and Wolfram [12]).

We believe that applications relevant to the industry needs in Houston area strongly motivate our students, increase their engagement in the learning process and enhance our lectures.

3. Sample Project: Secondary Oil Recovery.

When an underground source of oil is tapped, initially the oil flows out due to the high pressure in the reservoir. However, when the initial flow stops, there is usually a large amount of oil still left and secondary oil recovery process is used to extract more oil from the reservoir. This process assumes injecting water and/or gas through the injection wells which in turn pushes oil towards the production wells. $Such flow of water, gas and oil is described using the following system of partial differential equations

$$(s_w)_t + f(s_g, s_w)_x = 0,$$
$$(s_g)_t + g(s_g, s_w)_x = 0,$$

where x denotes one-dimensional spatial variable, t denotes time, s_w, s_g and s_o stand for saturations of water, gas and oil, and f and g denote flux functions. Students are given detailed explanation of the derivation of these equations as well as their basic mathematical properties.

Next, students are introduced to the Lax-Friedrich's finite difference numerical method for approximate solving of the above system of equations. The derivation of the method is explained in detail and it also compared to several other finite difference methods.

A bone structure of the computer code is implemented by instructor in Matlab and is provided for students to modify it for several initial value problems. A sample of the code is given below:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% MATLAB CODE FOR SYSTEMS OF CONSERVATION
% LAWS MODELLING OIL FLOW
%
% Look at the first example in the notes (paper "Numerical approximations of Riemann ...")
%
% here you need to set the initial data
%
x=5; t=1.5;     % x denotes space and t denotes time
%
% we create a mesh matrix and set the initial values
k=0; % initial time counter
    for r=1:1:T
        j=0; % initial space counter
        for c=1:1:X
        point(r,c).xx=j*dx/2; % space coordinate
        point(r,c).t=k*dt; % time coordinate
        if c==1
            point(r,c).p1=PL1;  % function P1
```

```
        point(r,c).p2=PL2;  % function P2
    elseif c==X
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Students are given instructions how to modify this code and how to use it to compute approximate solutions and obtain graphs for s_w, s_g and s_o for several initial value problems and at several different times. An example of the graphs is provided below in Figure 1. The injection well is located at the very left side of the graph and the production well is on the right. Clearly, we expect that water and gas injected at the injection well will push oil towards the production well. The students analyze these graphs and explain qualitative behavior of the approximate solutions.

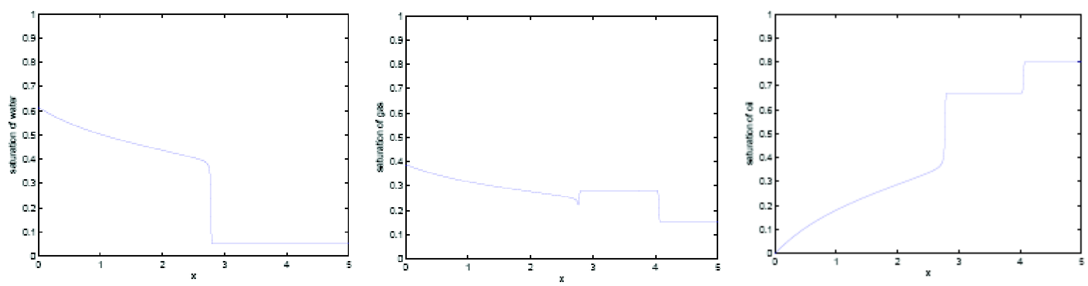

Figure 1. Graphs of approximate solutions for saturations of water, gas and oil.

During the implementation of this proposal within the Math 3301 course in the spring semester of 2009, a team of students who worked on the oil flow project were all computer science major. They actually wrote the entire code in C++ and used various features that C++ offers to make the code much faster and more efficient.

4. Conclusion.

The main goal of the above outlined project-oriented course is to provide an environment in which the students learn partial differential equations by actively working on research projects. Some of the learning strategies incorporated in the project are: group work, faculty-student research projects, active, cooperative and collaborative learning, learning by teaching others, teaching with technology, development of communication and analytic skills, student choices in class activities, in-class presentations, etc. Students do not learn much just by listening and taking notes; they must be actively engaged in learning, they must talk and write about what they are studying and (especially, since the field of differential equations allows it) they must relate it to the real world applications. This research component enhances course effectiveness and delivers an efficient learning experience.

The emphasis is placed on development of students' exploration and creative and higher-order thinking (using the technology, analysis, applying the theory, evaluation, etc.). Through such instructional methods, the project helps transition from the undergraduate classroom to the work/graduate school environments and builds a program that features extended and challenging opportunities for scientific growth. The students learn basics of Matlab, Maple or Mathematica, as well as of Microsoft Word and Power Point. Since the students work on the real world applications, we believe that the students are more motivated and engaged in the learning process. Unlike the traditional course delivery, the students are not only the passive recipients of the course material, but they discover knowledge by themselves. Furthermore, since the students are grouped in small teams, it is impossible for students to avoid participation.

Acknowledgments. The implementation of the above mentioned project-oriented courses was supported by the QEP (Quality Enhancement Program) Grant at the University of Houston – Downtown during the spring and fall semesters of 2009 and the spring semester of 2011. The author is grateful to Dr. Linda Becerra (UHD) for suggestions and advice while preparing the QEP proposal.

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