APPROXIMATION OF ROOTS OF EQUATIONS WITH A HAND-HELD CALCULATOR

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I. Introduction

As teachers and researchers in mathematics we often need to solve equations. The linear and quadratic equations are easy. There are formulas for the cubic and quartic equations, though less familiar. There are no general methods to solve the quintic and other higher order equations. There are a host of numerical solutions to solve these equations and other nonlinear equations as well. The disadvantage is that we need to relearn these methods and we need to program these algorithms in our computers before they can be implemented.

But oftentimes, we only need approximate solutions, and we need them fast, without recourse to programming the problems into our computers, or running them through our spreadsheets. We just need quick, simple, but nontrivial solutions to our equations. Here is where the hand-held calculator is most useful. With it, we can approximate roots of polynomial and nonpolynomial equations alike, using readily available algorithms derived from well-known classical methods. In this talk, we will discuss the bisection algorithm, the linear interpolation, Newton's method, and Horner's method, time permitting. The error analysis for these methods is also available that we can approximate roots to any predetermined level of accuracy. In this talk we will provide some examples on the use of several approximation methods implemented on the hand-held calculator. We will just use a simple scientific calculator for calculation. A graphing calculator would be an added bonus for a quicker solution, but not required in the following. We will close the talk with some discussion of the rates of convergence and error estimates for each of the methods used, and make a passing comment on complex roots, when they exist.

II. Classical methods

A. The bisection method

Let the function $y = f(x)$ be continuous on an interval containing *a* and *b, a* < *b,* and such that $f(a) \cdot f(b) < 0$. Then, $f(x) = 0$ has at least one solution *c,* $a < c < b$. That is, we know from the intermediate-value theorem that a continuous function has a root where the *y*-value changes sign from *a* to *b*; in particular, an odd-order polynomial always has at least one real root.

We decide in which half of the interval [*a, b*] the root lies by computing *f* at the midpoint of the interval, and picking that intermediate point where it differs in sign from either *a* or *b*. After *n* iterations, the root *c'* is within the interval $[a_n, b_n]$, where the error from the exact root *c* is $E = |c' - c| < \frac{b-a}{2^n}$. The number of approximating steps *n* is taken to be the smallest integer that satisfies the inequality:

$$n > \log_2\left(\frac{b-a}{E}\right) = \frac{\log\left(\frac{b-a}{E}\right)}{\log 2}.$$

Example: Find the root of $f(x) = x^4 - 4x^3 + 12$ to an accuracy of 0.01. In this example, $f(1) = 9, f(2) = -4,$ so that *a* = 1, *b* = 2, and *E* = 0.01. Then $n \sim 6.6 \sim 7$. The 7 iterations are:

$$f\left(\frac{3}{2}\right) = 3.5 \ldots > 0, \quad c' \in \left[\frac{3}{2}, 2\right]$$

$$f\left(\frac{7}{4}\right) = -0.58 \ldots < 0, c' \in \left[3/2, 7/4\right]$$

$$f(13/8) = 1.8 \ldots > 0, c' \in [13/8, 7/4]$$

$$f\left(\frac{27}{16}\right) = 0.88 \ldots > 0, c' \in \left[27/16, 7/4\right]$$

$$f(55/32) = 0.41 \ldots > 0, \quad c' \in [55/32, 7/4]$$

$$f\left(\frac{111}{64}\right) = 0.18 \ldots > 0, \quad c' \in \left[111/64, 7/4\right].$$

And $n = 7: \quad c' = \frac{223}{128} = 1.742 \ldots$; therefore, *c* = 1.74, to within 0.01.

B. Linear interpolation

Given the same conditions as in A above, if we connect the points $(a, f(a))$ and $(b, f(b))$ by a straight line, then the x-intercept of the line is a close approximation to the root c of the equation $f(x) = 0, a < c < b.$

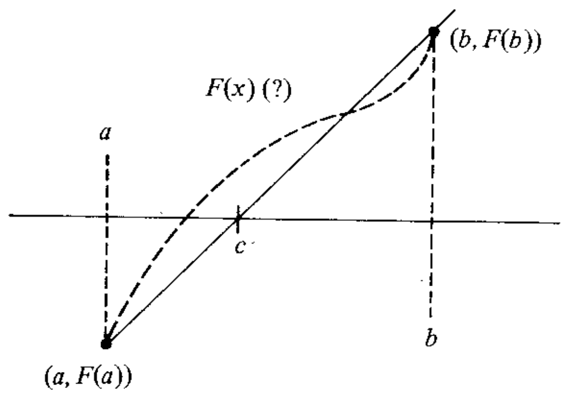

Figure 1. Linear Interpolation

By similar triangles, we find c:

$$\frac{c-a}{|f(a)|} = \frac{b-c}{f(b)}$$

or,
$$c = a + \frac{|f(a)|}{|f(a)| + |f(b)|}(b-a).$$

This linear interpolation is also known as the method of *regula falsi* (false position), since the straight line is not truly the graph of $f(x)$. The process may be iterated by using c as the endpoint of a new interval, where a new line may be drawn and a new x-intercept found. Each new interval usually adds a new decimal place to the approximation.

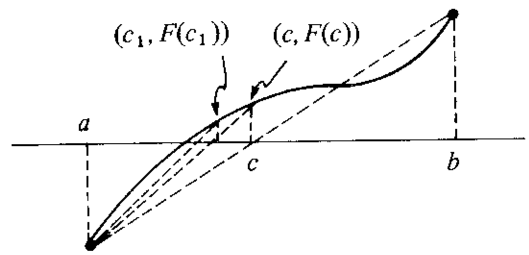

Figure 2: The method of *regula falsi*

Example: The same example above, accurate to within 0.001.
Here, $(a, f(a)) = (1,9)$ and $(b, f(b) = 2, -4)$. Then

$$c = 1 + \frac{9}{9+4}(2-1) = 1.692 \text{ , } c_0 \sim 1.7 \text{ , } c_0 \in [1.7,2];$$

$$f(1.7) \approx 0.70, f(2) = -4$$

$$c_1 = 1.7 + \frac{0.70}{0.70+4}(0.3) = 1.745 \ \rightarrow \ |c_1 - c_0| \approx 0.05, c_1 \in [1.7,1.75];$$

$$f(1.7) \approx 0.70, f(1.75) = -0.059$$

$$c_2 = 1.7 + \frac{0.70}{0.70+0.059}(0.05) = 1.746 \ \rightarrow \ |c_2 - c_1| \approx 0.001, \qquad c_2 \in [1.746,1.747].$$

Since $f(1.746) \approx 0.0026$ and $f(1.747) \approx -0.013$, and $|f(1.746)| < |f(1.747)|$, we take our approximate root $c = 1.746$, correct to within 0.001.

Remark: A straightforward approximation of the root may be obtained by successive subdivision of the interval [*a, b*], but one must be prepared to do a great deal of calculations [Sullivan, 2008]. Using the example above, $f(x) = x^4 - 4x^3 + 12.$
Here, $f(1) = 9, f(2) = -4,$ with $a = 1$ and $b = 2$.
Now, subdivide the interval [1, 2] into 10 equal intervals and calculate *f* at each new endpoint:

$f(1.1) = 8.1401$
$f(1.2) = \cdots$
$f(1.3) = \cdots$
$f(1.4) = \cdots$
$f(1.5) = \cdots$
$f(1.6) = \cdots$
$f(1.7) = 0.7001$
$f(1.9) = -0.8304$ Stop! The root lies in [1.7, 1.8].

Subdivide this interval and calculate *f:*

$f(1.71) = \cdots$
$f(1.72) = 0.3983$
$f(1.74) = 0.0943$
$f(1.75) = -0.0586$ Stop! The root lies in [1.74, 1.75].

Subdivide this interval again and calculate *f:*

$f(1.745) = 0.0179$
$f(1.746) = 0.00261$
$f(1.747) = -0.01268$ Stop! The root is 1.746, correct to within 0.001.

C. Newton's method

The two previous methods depended on two values of f at two values of x, and these values must be opposite in sign. Newton's method depends on one value of f and one value of its derivative f' at the same value of x.

The equation of the tangent line to the curve $y = f(x)$ at $x = a_0$ is given by:

$$y - f(a_0) = f'(a_0)(x - a_0).$$

The x-intercept of the tangent line is taken as the approximation to the root α of $f(x) = 0$:

$$x = a_0 - \frac{f(a_0)}{f'(a_0)}.$$

Assuming the derivative does not vanish, we may iterate the procedure and get an improved approximation:

$$a_{i+1} = a_i - \frac{f(a_i)}{f'(a_i)}.$$

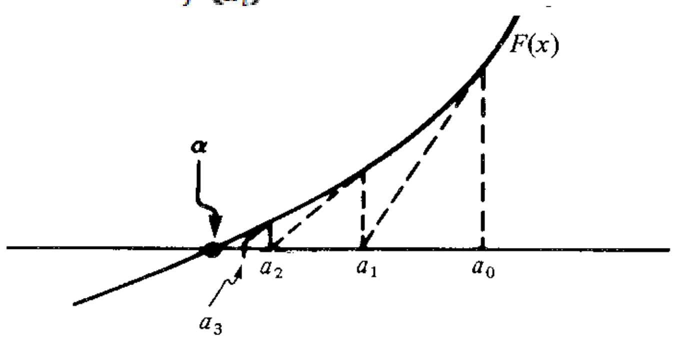

Figure 3: Newton's method

Example: Find one root of $f(x) = 2x^3 + 15x^2 - 57x + 30$ to an accuracy of 10^{-8}. The function has 3 real roots; we will just find that one in the interval [2, 3], that is, we know that $f(2) < 0$ and $f(3) > 0$. Let the initial approximation be $a_0 = 2.5$. Write the iteration formula as $G(x)$, and simplify:

$$G(x) = x - \frac{f(x)}{f'(x)} = x - \frac{2x^3 + 15x^2 - 57x + 30}{6x^2 + 30x - 57}$$
$$= \frac{4x^3 + 15x^2 - 30}{6x^2 + 30x - 57}.$$

This method has a quadratic convergence, that is, the number of accurate decimals doubles at each iteration, so that we only need to iterate 4 times:

$$
\begin{aligned}
a_0 &= 2.5 \\
a_1 &= G(2.5) = 2.27477478 \\
a_2 &= G(2.3) = 2.24092364 \\
a_3 &= G(2.24) = 2.23844071 \\
a_4 &= G(2.2384) = 2.23843900.
\end{aligned}
$$

The approximation is accurate to 8 places for:

$f(2.23843900) = -0.0000038$ and $f(2.23843901) = 0.00000065$.

And the solution is, therefore, $\alpha = 2.23843901$, which gives an f closer to 0 than the other value.

Remarks: (i) Although our examples were polynomial equations, all three methods really work for both polynomial and nonpolynomial equations.
(ii) The first two methods require the root to be isolated by a change of sign in f; Newton's method does not need this, but f must be differentiable. There are other subtle requirements for f for the method to work, but for most functions encountered, Newton's method works.
(iii) The accuracy of the methods discussed can be arbitrarily set and is mostly limited by the patience of the worker or the display capacity of the calculator used.

III. Convergence and error

The rates of convergence for the classical methods illustrated above are well-known. For the bisection method, the error at the nth step can be no more than $(b-a)/2^n$, where $[a, b]$ is the original interval containing the solution c. The speed of convergence is represented by the logarithmic expression for n, the number of steps needed to yield the prescribed accuracy.

For the linear interpolation, the rate of convergence is shown to be 'linear', that is, the error at one step is roughly a constant multiple (between 0 and 1) of the error at the previous step. This is considered slow convergence. In contrast, Newton's method has a 'quadratic' rate of convergence, that is, the number of correct number of decimal places doubles at each step. This is considered rapid convergence. Error estimates for both the linear interpolation and Newton's method have been found by advanced calculus. Expressions for error bounds may be found in the references, as well as other more refined techniques of approximating the roots.

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V. On complex roots

There are a number of standard methods for dealing with equations with complex roots. But when the polynomial has exactly one pair of conjugate roots, then the method is straightforward. We illustrate with an example [Dobbs and Hanks, 1992].

Example: Find all the roots of $f(x) = 6x^4 - 7x^3 + 25x^2 + (10 + 5\sqrt{2})x - 25$ to an accuracy of 10^{-4}.

For the real roots, by Descartes' rule of signs, we know that f has 1 negative root and 1 or 3 positive roots. Using the derivative of f, we can show that there is only 1 positive root:

$$\begin{aligned} f'(x) &= 24x^3 - 21x^2 + 50x + 10 + 5\sqrt{2} \\ &= (24x - 21)x^2 + 50x + 10 + 5\sqrt{2}. \end{aligned}$$

This expression is always > 0 for $x > 0$, so that f has only 1 positive root. Thus, the degree 4 polynomial $f(x)$ has 2 real roots (1 negative, 1 positive) and 2 complex (conjugate) roots.

Using any previous method discussed, the real roots are $c_1 = -1.0496$ and $c_2 = 0.7345$, correct to 4 decimal places. To find the complex roots:

$$f(x) = (x + 1.0496)(x - 0.7345) \cdot g(x)$$

By division, $g(x) = 6x^2 - 8.8906 + 32.4270$,
whose roots are: $c_{3,4} = 0.7409 \pm i2.2035.$

Write f in factored form:

$$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4),$$

where the α_i are the true roots. Then

$$f(c_4) = (c_4 - \alpha_1)(c_4 - \alpha_2)(c_4 - \alpha_3),$$

or,

$$|c_4 - \alpha_4| = \left|\frac{f(c_4)}{(c_4 - \alpha_1)(c_4 - \alpha_2)(c_4 - \alpha_3)}\right|.$$

Compute $f(c_4)$ by DeMoivre's theorem, approximate $(c_4 - \alpha_i) \approx (c_4 - c_i), i = 1,2,3,$ and get:

$$|c_4 - \alpha_4| = \left|\frac{-0.0086 + i0.0006}{(1.7906 + i2.2035)(0.0057 + i2.2035)(i4.407)}\right| = 0.0003,$$

with the same error estimate for $|c_3 - \alpha_3|$.

IV. Conclusion

When we encounter equations whose roots cannot readily be found, numerical methods are always available. And when we want fast, reliable, and nontrivial methods, the hand-held calculator can be very useful, suitable for polynomial and nonpolynomial equations alike. The techniques discussed here are quick and simple, and do not require any programming, with a minimum of button pushing. Their rates of convergence are rapid enough, and the order of precision can be set to any desired accuracy. Their ready availability is their most attractive feature.

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