MOVING RESULTS IN PLANE GEOMETRY AND COMPLEX ANALYSIS VIA GEOGEBRA

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Abstract

Two geometric figures coincided for Euclid when one could be moved to fit precisely on top of the other. This is how Euclid 'proved' SAS for triangles. Yet 2000 years passed before the idea of motion was properly introduced into geometry. Today, using the software GeoGebra, one can present moving geometric figures that motivate, suggest proofs, and deepen student understanding in subjects like plane geometry and complex analysis. We outline how GeoGebra can be used to illustrate the classification of isometries in the plane, including the Three Reflections Theorem. We also use GeoGebra to illustrate many of the properties of inversion 1/z in the complex plane.

Introduction. The mathematics software GeoGebra is freely available and allows the user to create interactive, educational mathematics demonstrations that dynamically unite geometry, algebra, and calculus.

Using familiar geometric quantities like points, lines, segments, angles, and vectors, one can create geometric objects and then dynamically change and modify them after their creation. This is very simple to do using a mouse and GeoGebra's toolbar.

One can also directly input expressions in algebraic notation, as well as symbolically differentiate and integrate a range of simple functions. All these can be combined together as well — GeoGebra allows one to view objects and inputs dually as expressions in an algebra window and as objects in a geometry window.

Indeed, GeoGebra may be an *electronic* version of the philospher René Descarte's dream. Descarte (1596 - 1650) wrote in his seminal work *Discourse on Method:*

> Those long chains of reasoning, each step simple and easy, which geometers are wont to employ in arriving at even the most difficult of their demonstrations, have led me to surmise that all the things we human beings are competent to know are interconnected in the same manner, and that none are so remote as to be beyond our reach or so hidden that we cannot discover them.

Descarte wanted to combine the fluid, visual, dynamic aspects of geometry with the powerful, symbolic, almost mechanical way to express complicated ideas with algebra. In his *Discourse* he wrote

> In this way I should be borrowing all that is best in geometry and algebra, and should be correcting all the defects of one by the help of the other.

It is useful to distinguish between what could be called the declarative and the imperative points of view. Mathematics frequently takes a declarative view of things. For example, one can prove that a unique (positive) square root $\sqrt{x}$ of a positive number x exists. But the more imperative viewpoint asks, 'How do you find $\sqrt{x}$?' This must be done algorithmically and constructively.

The corresponding declarative view of geometry uses the axiomatic method and deduction to prove theorems and to show that various contructions exist. GeoGebra allows one to take the imperative, constructive view and actually **build** things, move figures around, rotate and reflect objects. In doing this students can hone intuition and **play**. None of these activites is a substitute for rigorous mathematical proof, but letting students construct things with GeoGebra sows the seeds for proof ideas and illuminates the *need* for proof. This might inspire students to take the next steps toward mastering proof techniques and proving things for themselves.

Isometries and GeoGebra. We give some of the basic results for isometries of the plane. There are several different types types of isometries possible in the plane, and GeoGebra has many of them built into its toolbar. We use GeoGebra to

- show that each point in the plane is determined by its distances from three non-collinear points
- show that an isometry in the plane is determined by its action on three non-collinear points
- illustrate the idea of a direct and opposite isometry
- show that reflections are the building blocks for every isometry of the plane
- motivate the Three Reflections Theorem which says that every isometry in the plane can be built using at most three reflections

First, let's show that each point in the plane is uniquely determined by its distance from three points not in a line.

Two points A, B determine a unique line passing through them, but the same two points A, B also determine the unique line of points equidistant to A and B. This equidistant line is the perpendicular bisector of the line segment joining A to B and plays an important role in this proof that each point in the plane is uniquely located by its distances from three noncollinear points.

To show this uniqueness, suppose that there were two points P_1, P_2, both the same distances respectively from three noncollinear points A, B, C. This means that each of A, B, and C must lie on the equidistant line of P_1 and P_2. But this is not possible since A, B and C are noncollinear points. Thus P_1 and P_2 must coincide and we conclude that each point in the plane is uniquely determined by its distance from three points not in a line.

Next, we show that any isometry in the plane is uniquely determined by its action on three noncollinear points. Suppose f is the isometry, points A, B and C are not collinear, and we know $f(A), f(B)$ and $f(C)$.

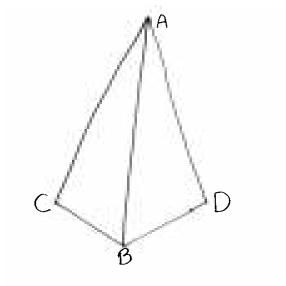


Figure 1: Isometries are determined by their actions on a triangle

Reflect the point C across the line AB so that line segment CD is perpendicular to line segment AB (see Figure 1). We use this construction to produce the segment CD which is perpendicular to AB. It follows that C is not on the line AB because line AB is the equidistant line for points C and D, and obviously C is not equidistant from C and D.

This argument depends only on the distances between the points A, B, C, and D, and these distances are preserved under mapping by any isometry. This means that if isometry f maps noncollinear points A, B, and C into $f(A), f(B)$, and $f(C)$, then these latter points are also noncollinear.

If Q is any point, we know that Q is uniquely determined by its distances from A, B, and C. Its image $f(Q)$ has the same respective distances from $f(A)$, $f(B)$, and $f(C)$, and so $f(Q)$ is uniquely determined as well. This means that any isometry which agrees with f on the points A, B, and C, agrees with f on any point Q.

Here, then, are some

Basic Results for Isometries of the Plane

- An isometry is a one-to-one and onto distance preserving map of the plane to itself.
- There are four basic types: translation, rotation, reflection, and glide reflection.
- The composition of two isometries (the 'product') is also an isometry.
- Isometries can be classified as direct and opposite isometries.
- Direct isometries (translations and rotations) preserve orientation.

- Opposite isometries (reflections and glide-reflections (a reflection composed with a translation)) reverse orientation. (Orientation can be thought of as the sense of rotation of angles.)

- Reflections are the fundamental building blocks: any isometry can be written as the product of reflections.

- The Three Reflections Theorem says: every isometry of the plane can be written as the product of three or fewer reflections.

- A point p is a fixed point of an isometry if it doesn't move when the isometry is applied to p.

- Isometries can be classified by the number of fixed points and whether they are direct or opposite. There are two possibilities for fixed points: zero, and one or more. And there are two types of 'orientations' for isometries: direct and opposite. It turns out, then, that are four types of isometry. We list them:

  - 1) Translations: zero fixed points, direct.
  - 2) Rotations: one fixed point, direct.
  - 3) Reflection: Infinitely many fixed points, opposite.
  - 4) Glide reflection: zero fixed points, opposite.

Complex Inversion and GeoGebra. The second slightly more advanced part of this paper illustrates the complex inversion map $z \mapsto 1/z$. This map is fundamental in the theory of Möbius transformations and has a number of surprising geometric properties. GeoGebra can be used to dynamically to illustrate the complex inversion map's action on lines and circles in the complex plane.

Here, then, are some

Basic Results for Complex Inversion $z \mapsto 1/z$

- The map $z \mapsto 1/z$ is called complex inversion, and its domain consists of $\mathbf{C} - \{0\}$. If we admit the point ∞ and allow $0 \longleftrightarrow \infty$, the complex inversion map is one-to-one and onto the complex plane $\mathbf{C}$.

- Lines not passing through 0 map to circles passing through 0.

- Circles not passing through 0 map to circles not passing through 0.

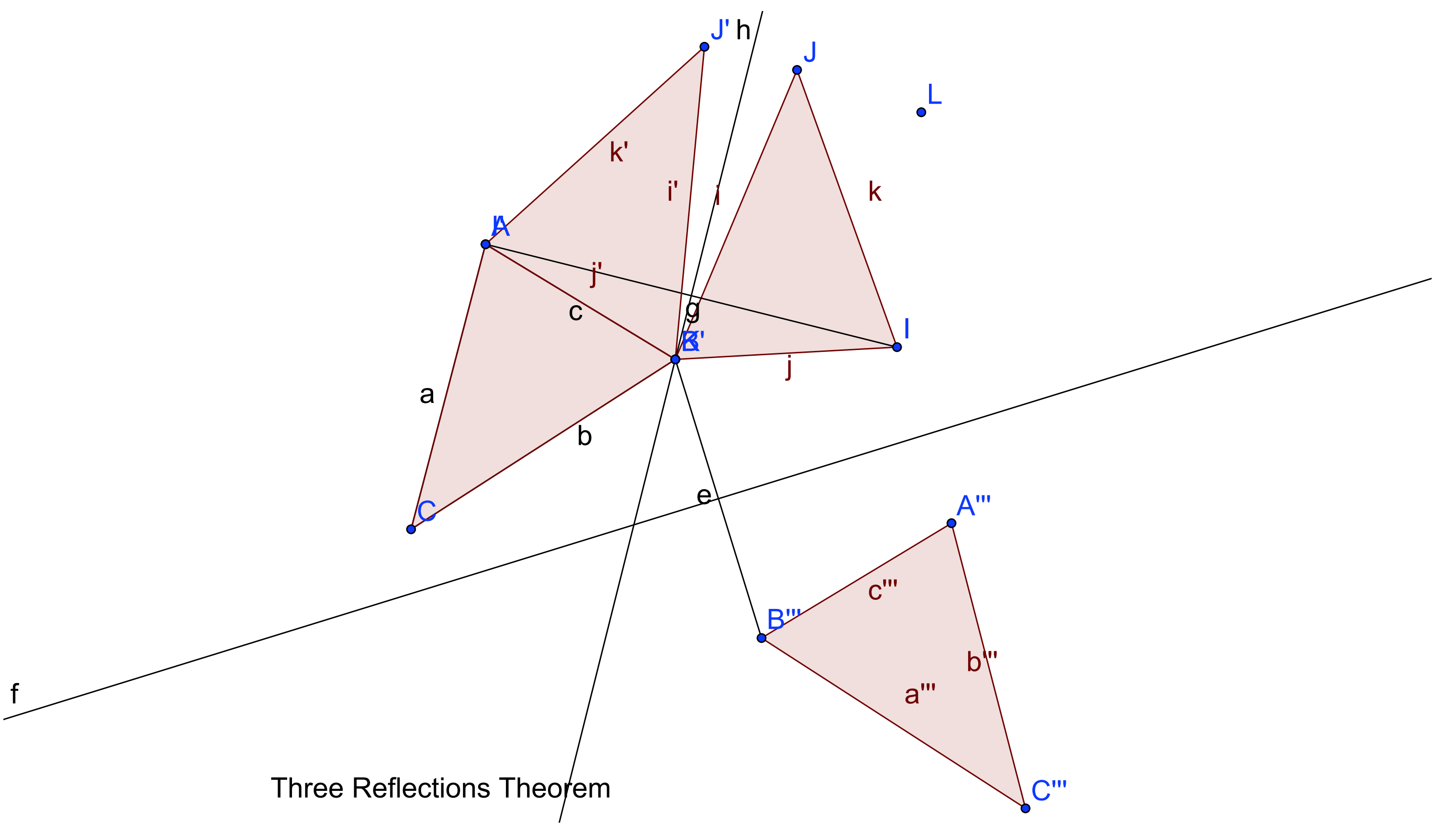
- Circles passing through 0 map to circles passing through 0.

- Lines passing through 0 map to circles passing through 0.

- It is convenient to think of a line as a circle of infinite radius with center at ∞. Then we have: under complex inversion, circles map to circles (much simpler!)

In the GeoGebra screen captures which conclude this paper, you will see an illustration of the Three Reflections Theorem in the plane in which one triangle is mapped onto another congruent to it using three reflections. You will also see an illustration of the complex inversion map and its action on lines and circles in the plane. Of course the dynamic aspects of GeoGebra are lost in this presentation!

Have fun using GeoGebra!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Three Reflections Theorem

This demonstration demonstrates complex inversion,
z mapping to 1/z in the complex plane.

In the demonstration, Z ' = 1/Z and Y ' = 1/Y.

The dotted circles are the loci of Z and Y as they move
respectively along a circle and a line.