USING ANIMATIONS TO VISUALIZE ABSTRACT CONCEPTS

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Abstract

Mental models provide mathematicians with ways to visualize, at least mentally, abstract constructs of mathematics. We consider how animations in electronic media can be used to provide undergraduate online students with usable mental models and investigate how learners use these models in their reasoning.

Formal Reasoning, Proofs and Definitions

The transition from lower-division courses to upper-division courses often provides a substantive hurdle to students. Lower-division courses, like Calculus, Differential Equations and the First Course in Linear Algebra often emphasize routine computations, while upper-division courses emphasize the use of formal definitions and formally proving Theorems. Courses in this category are Abstract Algebra, Real Analysis and Point Set Topology, to name a few. The transition from lower-division courses (i.e., Calculus, Differential Equations and the first course in Linear Algebra) to the advanced courses poses a difficulty for many students: The concept of proof and formal definition has been marginalized in both the high-school curriculum and the first years in college (Moore, 1994; Wu, 1996).

Yet the importance of proof and formal reasoning is widely recognized. The National Council of Teachers of Mathematics standards of 2000 (NCTM, 2000) declare: "*By the end of secondary school, students should be able to understand and produce some mathematical proofs--logically rigorous deductions of conclusions from hypotheses--and should appreciate the value of such arguments.*" In order to prepare students for the challenges of the upper-division courses, many colleges now include a "Bridge" course in their curriculum. The purpose of this course is to acquaint the student with general proof techniques and the basics Analysis, Abstract Algebra and other advanced subject areas.

Mental Models

Students often attempt to create mechanically a proof by randomly applying definitions and theorems. This attempt of course is futile. Key to developing in an understanding of the abstract reasoning that underlies the mathematical proof is to be able to associate meaning with the formal definitions. As Rav (1999) notes, "...writing on the board a formal definition without detailed explanations of the intended meaning is a sure way to block comprehension." Tall (Tall,

1999) recognizes that a students' cognitive structure must undergo a transition from describing concepts verbally to using verbal definitions that prescribe the concepts.

In order to be able to create a correct proof, students must understand the definition well enough in order to use it in establishing a theorem – that is, students must develop a "workable definition" (Bills & Tall, 1998). While not directly referencing the concept of workable definitions, Alcock and Simpson (Alcock & Simpson, 2004; Alcock & Simpson, 2005) consider how students associate meaning to abstract definitions. They consider two groups of students – visualizing and non-visualizing – and describe ways in which these students associate meaning with a formal definition. Students in the visualizing category use diagrams appropriately to get guidance in formulating a proof. Their mental model of mathematical concepts appears to be general enough so as to anticipate counterexamples. For example, these students use a mental image of a sequence of real numbers that is fluid: The mental image of "sequence" is one that includes convergent, oscillating, and unbounded sequences. Students in the non-visualizing category appear to treat mathematical statements as a unified object. In particular, they appear to understand the correct use of quantifiers.

The need to help students visualize abstract concepts is as important in traditional face-to-face classes as it is in online classes. Dynamics of online classes have been studied since the mid-1990's and the consensus of best practices appear to include creation of a community of learners, course material in addition to a well-chosen textbook and frequent, meaningful feedback (for example, Kirschner & van Bruggen 2004).

Animations and Mental Models

In the following section we will provide examples how instructor-created course materials can help students make sense out of definitions. The examples were generated using Microsoft Office's PowerPoint: The instructor created animations in PowerPoint and recorded narration to explain concepts in courses such as Real Analysis and Topology. Figure 1 shows how narration and animation within a single PowerPoint slide bring the concept of convergent sequence to life.

Let us recall that a sequence (s_n) is convergent to a limit s if and only if for every $\varepsilon > 0$ there exists an N such that for all natural numbers $n > N$, $| s_n - s | < \varepsilon$. This definition has three quantifiers (is an $\forall\exists\forall$-statement) and the order of the quantifiers is important. The sequence of events in the animation suggests the order in which the quantifiers must be considered: $\forall\varepsilon>0$ first, then $\exists$ N, and finally $\forall$n>N.

The slide also offers two different scenarios for different choices of ε. It therefore suggests (and the narration makes it explicit), that we must be able to carry out the process of finding a suitable N for every possible choice of ε.

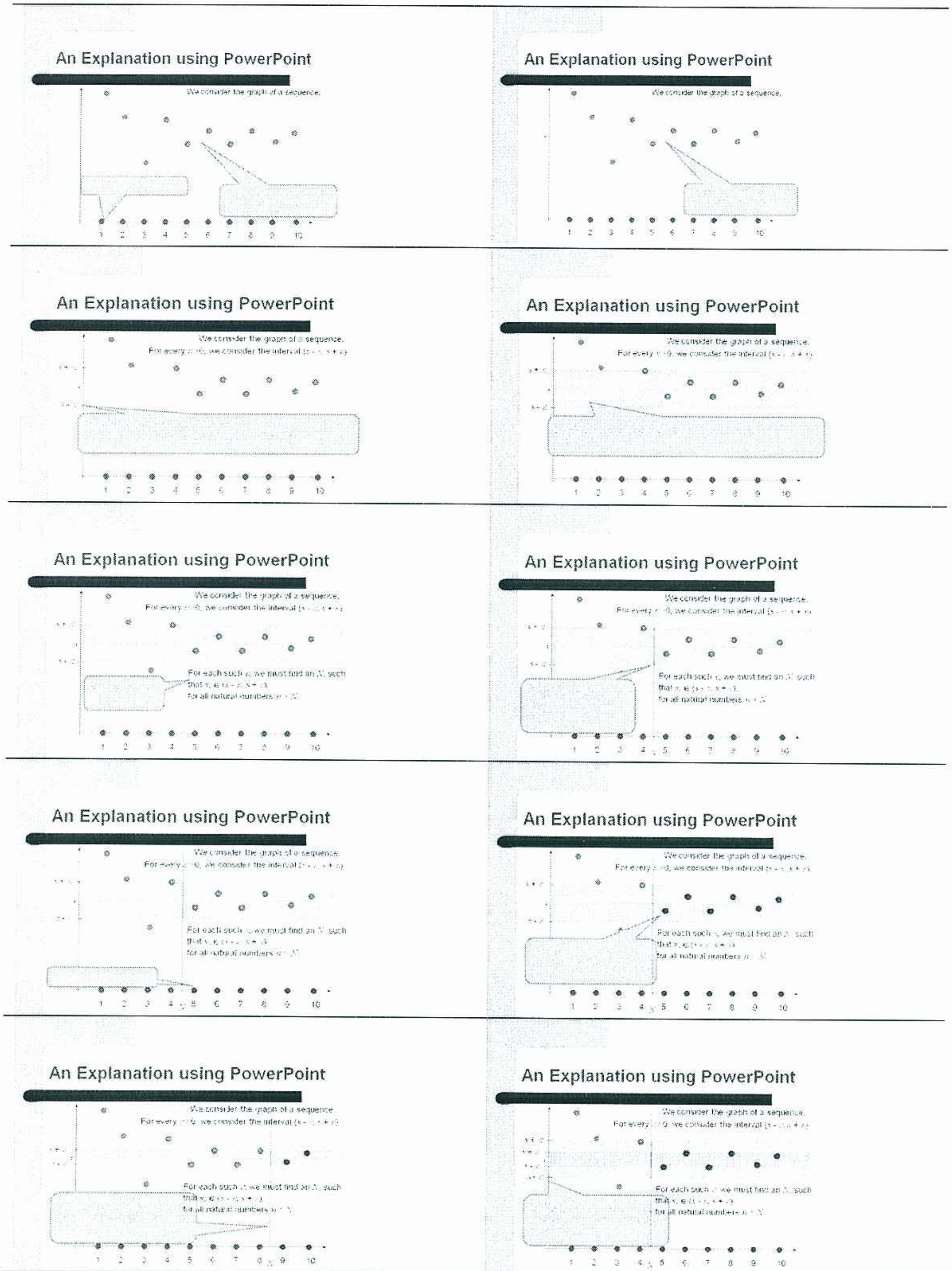

Figure 1: Progression of Animation on one PowerPoint Slide

The use of animated diagrams is the key tool in this course concept that helps visualizing students gain an understanding of an abstract concept. In contrast to a printed version of the same diagram, an animation allows the viewer to visualize the order of the quantifiers. We now consider some examples of student work and student contributions to the discussion forum. Many of the comments clearly indicated that students associated some pictoral meaning with the symbols. The write:

> "We can squeeze s_n close to s, if we choose a large enough N..."
> "The new N just moves further out along the line."
> "ε/k is just another small ε, so we can find a N such that ..."

While these statements are far from being precise and correct statements in the mathematical sense, they indicate that students view the definition of a convergent sequence as a fluid object: Terms of the sequence are being "squeezed" so that they are closer to the limit and the value of "N" moves along the line.

The statement that "ε/k is just another small ε" arose in the discussion of the proof where students were asked to prove that the sequence $(k\, s_n)$ converges to $k\, s$, where s is the limit of the sequence (s_n). It is clear here, that the student understands the meaning of the universal quantifier in the definition and infers correctly that any positive value may be used for ε.

Some of the examples above still contain mistakes. For example, the student who referred to ε/k as being "just another small ε" later corrected this statement into $\varepsilon/|k|$, which indeed leads to the correct conclusion.

Further Questions

Further investigation should consider in depth the impact of animations in definitions on the formation of students' mental models and how students use those models in their reasoning.

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