PURPOSEFUL DRAGGING: MOTIVATING DEEPER MATHEMATICAL UNDERSTANDING THROUGH DYNAMIC GEOMETRY EXPLORATION

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Introduction: The Critical Role of Prototypes in Mathematics

At a recent conference we asked participants to quickly sketch examples of mathematical objects commonly encountered in courses they teach. As participants drew triangles, rectangles and quadratic functions, the vast majority of university mathematics faculty drew "prototypes" of each: an equilateral or right triangle; a rectangle with two longer sides parallel to the x-axis, and the function $y = x^2$. Rosch and Mervis (1975) write about prototypes as "those members of a category which most reflect the redundancy structure of the category as a whole" (p. 602). Alternatively, Schwartz and Hershkowitz (1999) describe prototypes as "the members of a category that have a set of features most highly correlated with the features of other members" (p. 363). While few would dispute instructional advantages associated with prototypes in the teaching and learning of mathematics, prototypical thinking encourages students to see tasks and concepts in predictable ways, too often obfuscating more creative alternatives. In this paper, we provide examples of ways in which we have exploited the tendency of students to think prototypically to deepen their understanding of various mathematical concepts. Through careful modification of routine classroom examples, we present our students with mathematical scenarios that yield non-prototypical results. As the examples illustrate, dynamic geometry software [DGS] has played a pivotal role in the creation of such tasks.

The Fence Problem: Re-envisioning a Prototypical Optimization Task in Calculus

In calculus, students routinely encounter prototypical functions and tasks - the box problem, related rates, and standard optimization problems to name but a few. Fencing problems are familiar optimization tasks to most calculus teachers. Variations of the following example are commonplace in any calculus text.

A rectangle plot of land is to be enclosed using exactly 100 feet of fencing. What dimensions will maximize the rectangular area?

As is the case with most optimization tasks, the solution of the preceding problem involves a standard, easily differentiated *polynomial* (a prototypical type of function). In addition, the area is maximized by enclosing a *square* plot of land (a prototypical shape). The tracing features of a DGS can be used to quickly generate a sketch of the area of a rectangle ABCD with a fixed perimeter with respect to its length (see Figure 1).

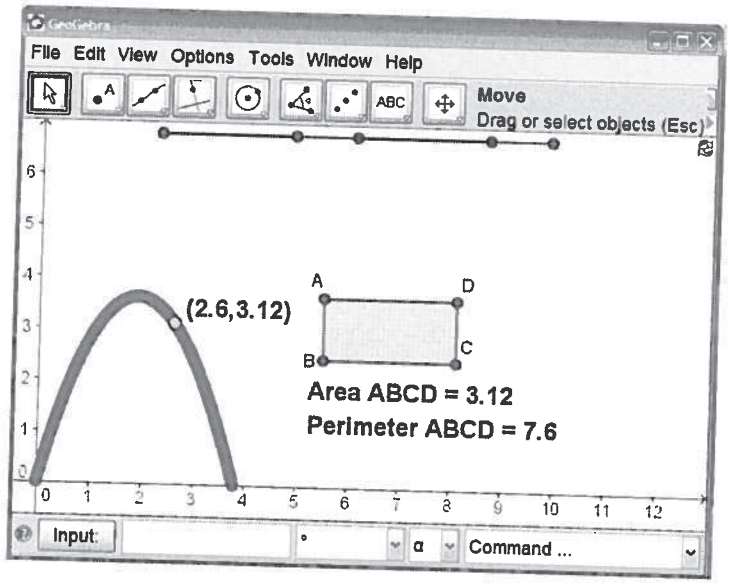

Figure 1. Sketch of a parabola, generated by prototypical fence problem with a fixed perimeter.

Because such solutions - namely, squares and parabolas - appear so frequently, students come to expect similar solutions in all problems they encounter. Tasks such as these and their solutions may be considered "prototypical." However, by twisting a prototypical problem slightly, new tasks are generated that encourage students to think more deeply as they explore the intricacies of their solutions.

> A rectangle plot of land has an area of 18 square feet. What dimensions minimizes its perimeter?

Students' prior experiences with prototypical tasks influence their responses to this revised task, so much so that the DGS may in advertently reinforce their prototypical thinking. As the students aimlessly drag the point to change the dimensions of the rectangle, a seemingly prototypical solution initially appears (see Figure 2).

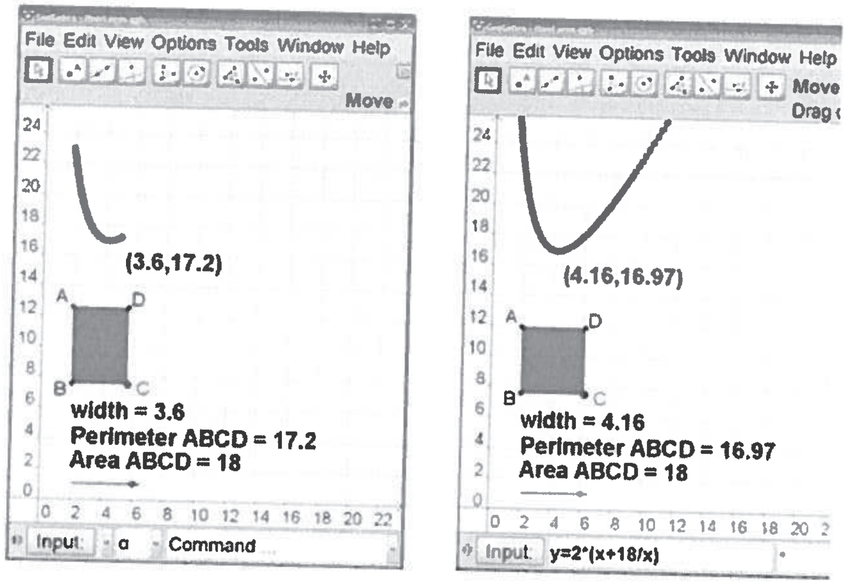

Figure 2. On the left, a non-prototypical fence problem with a fixed area; on the right, a graph of the non-prototypical function, $P = 2(x + \frac{18}{x})$.

Not surprisingly, students "see what they want to see," namely a parabola similar to the previous task's solution. However, with more thoughtful dragging, students soon realize that the function related to this task is not a prototype. Indeed, the plot of perimeter with respect to width is not fitted with a parabola. The unexpected result associated with the task encourages class discussions of the underlying algebra of the function as students rigorously verify conjectures generated with technology. For instance, students verify conjectures by using the area formula of a rectangle and solving for one of the sides, substituting the resulting expression into the perimeter equation yielding $P = 2(x + \frac{18}{x})$.

Geometrical Probability Tasks: A Prototype and an Unexpected Result

In *Fostering Geometric Thinking*, Driscoll et al (2007) discusses instructional advantages and disadvantages associated with prototypes in the study of geometry. In particular, he notes the tendency of students and teachers to look for regular shapes when solving problems, since a wide variety of tasks involve such shapes in their solutions. Consider, for instance, the *Triangles from Altitudes* task, a classic geometric probability problem.

***Triangles from Altitudes*:** A point P is chosen at random inside an equilateral triangle. Find the probability that the three perpendiculars from P to the sides of the triangle can be rearranged to form a triangle.

With DGS and rudimentary knowledge of the triangle inequality, students can construct a dynamic sketch to informally locate points within an equilateral triangle that yield perpendicular segments that form triangles. The set of all desired points in fact forms a prototypical shape, namely, a midpoint triangle with area equal to 1/4 the original (see Figure 3).

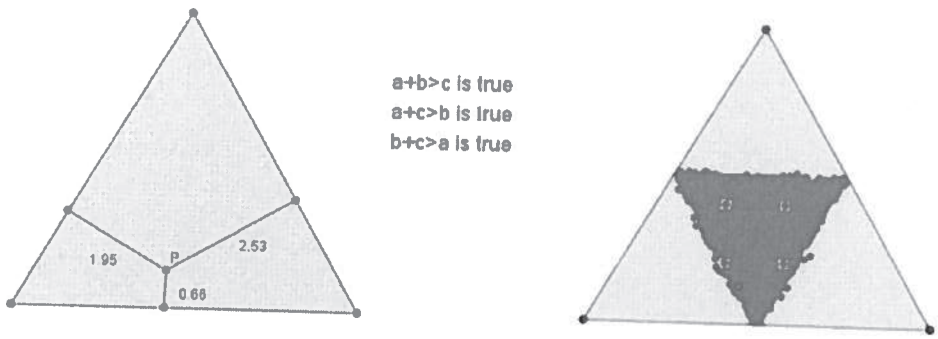

Figure 3. As students drag point P in the triangle on the left, the corresponding point is traced in the translated triangle on the right.

The solution to the *Triangles from Altitudes* task is prototypical in the sense that its solution may be represented as a familiar shape, namely an equilateral triangle. Moreover, the numerical solution is a familiar fraction, namely 1/4. For these reasons, it is interesting to pose the *Acute Angles* task as a follow-up.

Acute Angles Task. A point P is chosen at random inside an equilateral triangle ABC. Find the probability that one of the triangles ABP, APC, and PBC is acute.

While also a geometric probability problem (involving an equilateral triangle, no less), the solution to the problem is unexpected. Tracing points within the equilateral triangle that generate an acute angle may be modeled with DGS in much the same way as in Figure 3. In our experience, many students who solve these tasks simultaneously identify the complement of the solution set as an equilateral triangle; despite the apparent bounding arcs generated by the sketch (see Figure 4).

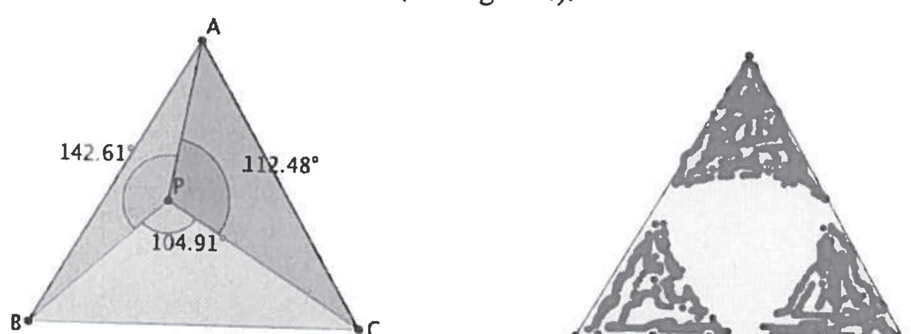

Figure 4. As students drag point P in the triangle on the left, the corresponding point is traced in the translated triangle on the right when at least one of triangles ABP, APC, and PBC is acute.

The solution, in fact, generates a Reauleaux triangle. Understanding of this result involves a deeper exploration of the task than blind dragging alone. Knowledge of Thales' Theorem – namely that a triangle inscribed in a semicircle forms a right triangle - is helpful to determine the actual probability, $\frac{\sqrt{3}-\left[\frac{1}{2}(\pi-\sqrt{3})\right]}{\sqrt{3}} \approx .59$.

Conclusion

At the most naïve level, students recognize mathematical entities as a collection of familiar examples. As they learn mathematics, students construct prototypes of mathematical objects and problems that in many instances are productive. However, prototypical thinking can interfere with creative thought. Teachers and textbooks unwittingly present prototypes to students without providing alternatives. In this paper we expanded on two familiar textbook examples, transforming them to tasks that generate non-prototypical solutions. If we wish to engender creativity and true problem solving in classes, we must present such examples with our students.

References

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