

# RIGHT OR WRONG? PROOF AND KNOWLEDGE IN AN ONLINE CLASS

Markus Pomper  
Indiana University East  
2325 Chester Blvd, Richmond, IN 47374  
mpomper@indiana.edu

## **Abstract**

Peer-review and discussions in an online real analysis course can help students write correct proofs. Students' assignments illustrate that they become more proficient as the semester progresses. More surprisingly, students' beliefs about their role within mathematics also changes: They develop the confidence to decide whether a proposed proof is correct.

## **1. Context**

Online courses have become the medium of choice for students who cannot otherwise attend in a traditional classroom setting. Online distance education allows students to take courses in the convenience of their home or office, while at the same time being provided with almost instantaneous access to the instructor and an abundance of online resources.

A critical observer would rightfully suspect that the lack of face-to-face interaction in an online class can be a significant detriment to learning. This is particularly true for upper-division classes, where teaching the concept of proof has long frustrated mathematics faculty. In the traditional sequence of subjects in mathematics, the proof concept is covered in Geometry class in high school, and is then marginalized until a student reaches Junior or Senior standing in college.

Numerous ways of teaching proofs in college have been proposed: Moore (1994) suggests a method by which students are presented with a statement and are then asked to prove the statement (or disprove it, in case it is false). Each proof suggestion is discussed in class and the course progresses only a convincing proof has been found. This method relies heavily on interaction between course participants. In the context of an online class, where students typically meet in an asynchronous setting, the Moore method requires some modification.

The method used in the course under discussion in this paper required students to use an online discussion forum to propose various attempts at proving given statements, and to critique (or correct) each others' attempts. Using the discussion as a basis, each student prepares a draft of a proof, which is then shared with a peer. It is the peer's task to review the proof, determine whether it is correct, and suggest ways in which the proof can be corrected or improved upon.

In the following section, we will consider several examples of student work and how the peer-review helped students hone their skills in proving. In the following section, we will consider the progress students made throughout the semester and discuss the students' epistemic beliefs.

## **2. Examples of Student Work**

In order to examine how students' ability to write proofs increases, we examine proofs from several stages of the semester.

The first example is from the beginning of class. Students were asked to prove the following statement:

*Suppose  $p$  and  $q$  are integers. Recall that an integer  $m$  is even iff  $m=2k$  for some integers and  $m$  is odd iff  $m=2k+1$  for some integer  $k$ . Prove the following: If  $p$  is odd and  $q$  is odd, then  $p+q$  is even.*

**Student 1**

Hypothesis: If  $p$  is odd and  $q$  is odd, then  $p + q$  is even

$p = 2k + 1$	$k$ is an integer, definition of odd integer
$q = 2k + 1$	$k$ is an integer, definition of odd integer
$p + q = (2k + 1) + (2k + 1)$	substitution
$p + q = 4k + 2$	combine like terms
$p + q = 2(2k + 1)$	distribution property

Conclusion:  $p + q$  an even number since  $2k + 1$  is an odd integer and by definition 2 times an integer is an even integer.

We observe that this proof attempt is incorrect because the student chooses the same representation for the integers  $p$  and  $q$ . The peer reviewer pointed out this problem, which allowed Student 1 to correct the mistake in his final draft.

We now consider a solution attempt of a second student:

**Student 2:**

If  $p$  is odd and  $q$  is odd, then  $p + q$  is even.

Hypothesis: $p + q$ is even.	
$p = (2k + 1)$ for some integer $k$	[definition].
$q = (2k + 1)$ for some integer $k$	[definition].
$(2k + 1) + (2k + 1)$	[adding $p$ and $q$ ]
$= 4k + 2$	[simplifying by addition]

Conclusion:  $4k$  will always give an even integer and adding 2 will give a new even integer.

This proof attempt also uses the same representation of  $p$  and  $q$ . In addition, the student does not distinguish between the symbol “+” and the logical connective “and”. This problem aside, the student does not realize that the conclusion “ $4k + 2$  is an even integer” is reached not by referring to the definition or to a previously proven statement, but simply to intuition.

This student’s peer reviewer pointed out the numerous flaws in the reasoning and provided several ways of correcting the proof. However, the final version of this student’s proof still did not use the definition of even integer in arriving at the conclusion.

It is also interesting to note that most students in the class choose a two-column approach to a proof. One might surmise that this format is reflective of a students’ belief that a valid proof must follow a certain format (e.g., the two-column proof from High-School Geometry).

We now fast-forward to mid-semester. The topic of under discussion are ordered field axioms. Students are asked to prove the statement: *If  $x \geq 0$  and  $x \leq \epsilon$  for all  $\epsilon > 0$ , then  $x=0$ .*

The possibilities of proving this statement were discussed extensively in the discussion forum and students concurred that a proof by contradiction would probably be the best attempt.

Again we examine two students' attempts.

**Student 1**

One way to do this is to realize that by Axiom O1 we know that one of the three cases must hold:  $x = 0$ ,  $x < 0$  or  $x > 0$ . The given consequent for this statement is that  $x = 0$ . In this instance we can show that the two cases of  $x < 0$  and  $x > 0$  do not hold and therefore  $x = 0$ .

Our Hypothesis is that  $x \geq 0$  and  $x \leq \epsilon$  for all  $\epsilon > 0$ .

In the case of  $x < 0$  we know this to be false because we have said in our hypothesis that  $x \geq 0$ . So  $x$  is  $\geq 0$ .

In the case of  $x > 0$ . Let  $\epsilon = x/2 > 0$ , so we know  $x/2 < x$  that but in the hypothesis we see that  $x > \epsilon$  and we have a contradiction of the statement that  $x > 0$ . So  $x$  is not greater than 0.

While clumsy, this student provides a correct proof of the statement. What is interesting here is that the student deviates from the two-column format. He also chose not begin with a contradiction, and rather use the Law of Trichotomy to begin his proof. This was not discussed in the discussion forum, and the student seems almost apologetic about this fact, stating "One way to do this...".

The peer reviewer recognized this proof as valid and recommended several suitable ways of reducing the length of the proof.

We do not reproduce the solution attempt of Student 2, whose draft consisted primarily of phrases from the discussion forum that had been cut and pasted together.

Student 3 had started the semester off very well, providing a very detailed and correct proof of the example discussed above. Up to this point the semester, she had relied on mechanically using the axioms and tautologies in order to generate proofs. We consider an excerpt of the proof attempt for this assignment:

**Student 3**

To prove the implication "If  $x \geq 0$  and  $x > \epsilon$  for all  $\epsilon > 0$ , then  $x = 0$ ." we will use the ordered field axiom O1 which says: For all  $x, y \in \mathbb{R}$ , exactly one of the relations  $x = y$ ,  $x > y$ , or  $x < y$  holds (trichotomy law). We will show that with the given antecedent, the consequent cannot be  $x < 0$  or  $x > 0$  and must therefore be  $x = 0$  by the trichotomy law. Thus, we will show the following implications to be contradictions:

Case 1: If  $x \geq 0$  and  $x \leq \epsilon$  for all  $\epsilon > 0$ , then  $x < 0$ .

Case 2: If  $x \geq 0$  and  $x \leq \epsilon$  for all  $\epsilon > 0$ , then  $x > 0$ .

Using the contrapositive of Case 2, we have: If  $x \leq 0$ , then  $x < 0$  or there exists an  $\epsilon > 0$  such that  $x > \epsilon$ .

...

Case 2: Suppose  $x \leq 0$

If  $\epsilon = (x/2)$ , then  $(x/2) > 0$  and  $x > (x/2)$ .

Thus  $0 < (x/2) < x$  and we have

$x > 0$  which is a contradiction.

substitution of  $\epsilon = (x/2)$ .

combining the two implications above

O2 – For all  $x, y, z \in \mathbb{R}$ , if  $x < y$  and  $y < z$ , then  $x < z$ .

This student does not appear to understand what the statement is saying, or why it might be true. She applies axioms and tautologies without truly understanding what the modified statement might imply, and whether it is easier to prove than the original statement. In this respect, the student has arrived at an impasse: Following a set procedure in trying to establish the truth of a statement is not a suitable course of action.

The peer reviewer concluded that this proof was confusing and "probably not correct". While the peer reviewer could not give any further advice, the instructor's critique provided the student with

suggestions how to correct the proof. The final version of this student's proof was correct, but still very clumsy.

We now consider an assignment from the end of the semester. Here, students were asked to prove or disprove the following statement: *Suppose that  $(s_n)$  is a sequence of real numbers. Prove or disprove:  $\lim s_n = 0$  iff  $\lim |s_n| = 0$ .*

**Student 1**

This statement is true. The conditional "If  $\lim s_n = 0$  then  $\lim |s_n| = 0$ ." had been proved in an earlier exercise. We therefore consider only the reverse implication in reproducing students' work.

Conversely we must show that if  $\lim |s_n| = 0$  then  $\lim s_n = 0$ .

If  $\lim |s_n| = 0$  we know that there for every  $\epsilon > 0$  there exists an  $N$  such that  $||s_n| - 0| < \epsilon, n > N$ .

This simplifies to  $||s_n| < \epsilon$

If the  $\lim s_n = 0$  then  $s_n$  must converge to 0 so we see for every  $\epsilon > 0$  there exists an  $N$  such that  $|s_n - 0| < \epsilon, n > N$ .

This simplifies to  $|s_n| < \epsilon$

We know that  $||s_n| = |s_n|$ .

And if for every  $\epsilon > 0$  there exist an  $N$  such that  $||s_n| < \epsilon$

We know that for that arbitrary value of  $\epsilon$  and the same  $N$ ,  $|s_n| < \epsilon$

We observe that this student's skill in providing formal proofs has evolved further. The student's work is less centered on emphasizing algebraic manipulation and focuses more on the use of definitions, in particular, on definition of limits of sequence. It is clear, that this student has built a clear mental concept of this definition, as he can make plausible the choice of the  $N$  in proving that  $\lim s_n = 0$ . We contrast this with a proof attempt of student 3:

**Student 3:**

On the other hand, suppose  $\lim |s_n| = 0$ .

Then by definition 16.2,  $||s_n| - 0| < \epsilon$  for all  $\epsilon > 0$ .

Notice that  $||s_n| - 0| = ||s_n| = |s_n|$

Since  $s_n = s_n - 0$  by addition axiom A4, we obtain  $|s_n| = |s_n - 0|$

Therefore,  $|s_n - 0| = ||s_n| - 0| < \epsilon$  for all  $\epsilon > 0$ ,

and we obtain  $|s_n - 0| < \epsilon$  for all  $\epsilon > 0$ .

By definition 16.2,  $\lim s_n = 0$ .

This student places heavy emphasis on the use of the field axioms, but fails to adequately employ the definition of the limits. In fact, the student claims that  $\lim |s_n| = 0$  is equivalent to  $||s_n| - 0| < \epsilon$  for all  $\epsilon > 0$ : She entirely misses the point of the quantified statement "for every  $\epsilon > 0$  there exists a  $N$ , such that for all natural numbers  $n \geq N, \dots$ "

**3. Right or Wrong?**

We now consider the possible interplay of Self-Authorship and students' perception in their role of proof construction and verification. The term Self-Authorship was coined by Baxter-Magolda (2001). Baxter-Magolda studied the epistemic belief systems of young adults. Through a series of interviews, beginning when the subjects were in their early 20s, and ending when they had reached their mid-30s, Baxter-Magolda concluded that these young adults' belief systems about knowledge and their own role in creating knowledge changed. When they were beginning their college career, they regarded knowledge as static: They expected their professors to provide knowledge and saw their role merely in absorbing what the professors would divulge. A person with this epistemic belief system may be described as a "user of knowledge". In later interviews Baxter-Magolda's subjects realized that knowledge is not absolute, that it is evolving and that

they played a role in forming knowledge. Baxter-Magolda calls these persons “creators of knowledge.” She termed the transition from user to creator the Crossroads. Little research has been done on what triggers a person to proceed to the Crossroads (Pizzolato, 2005). It is clear, however, that the process of transitioning from a user of knowledge to a creator of knowledge takes many years.

A discussion of the students’ self-disclosed backgrounds lets us gain some insight into their progression to Self-Authorship and gain an appreciation why these students improved (or did not improve) on their proof-writing ability:

Student 1 is in his late 40’s. After a career in engineering, this student wishes to gain a certification for teaching secondary level mathematics. From the information this student shared, we surmise that he has already transitioned through the crossroads. His proofs reflect confidence in his ability to discern a correct proof attempt from an incorrect one, without needing validation from the instructor.

Student 2 is in his early 20’s. He is working full-time and completing his degree in secondary education online. This student attempts to prove statements either through rote manipulation of symbols or by referring to intuition. His ability to write correct proofs and evaluate others’ proof attempts does not seem to progress.

Student 3 is in her late 20’s. She is already a certified secondary education teacher and takes this course for purposes of recertification. Her early proof attempts show a heavy reliance on following patterns and seeking external validation from the instructor. Judging from her postings on the course discussion forum and the feedback she provides to her peers, it appears that she does not see herself capable of assigning meaning to the formal statements, and much less, determining whether a statement is correct. Therefore, she is not able to recognize that one of the statements in her proof follows trivially from the axioms; she therefore fails to provide a complete argument. This episode appears to mark a turning point in Student 3’s epistemic beliefs. She attempts to build an understanding of the meaning associated with the symbols. The progress in doing so is slow, and throughout the semester she defers judgment regarding the correctness of arguments to the instructor. We therefore surmise that she is about to enter the Crossroads.

#### **4. Conclusion**

The present case study does not provide sufficient data about the interplay between self-authorship and students’ ability to create proofs and/or to verify the correctness of formal arguments. More research in this area might lead to interesting results.

#### **References**

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