

UTILIZING CAS TECHNOLOGY TO ENHANCE STUDENTS UNDERSTANDING OF THE COLLATZ PROBLEM

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ABSTRACT: In 1937, the German Mathematician Lothar Collatz (1910-1990) conjectured the following: Take any positive integer n . If n is even, divide by two while if n is odd, triple and add one. Apply this procedure on each new integer obtained. After finitely many steps, the sequence will converge to one. This problem over the years has taken on various names such as the $3X+1$ problem, Ulam's Conjecture, and The Syracuse Problem in deference to the work done by Stanislaw Ulam and Syracuse University in trying to resolve the problem or at least extend the upper bound for the verification of the conjecture. While this conjecture has been verified for each of the positive integers through at least $3 \cdot 2^{53}$, it remains unresolved to this day. This problem is easily posed, but like many in elementary number theory, the solution is far from simple. The fact that the problem remains unresolved is certainly not due to a lack of effort. More computer time has been spent in trying to obtain a foothold on this problem than any other. The purpose of this paper is to show how CAS technology can aid in understanding this problem and help one discover some neat mathematics. Among the activities developed will be obtaining a TABLE for this sequence deploying the SEQUENCE MODE on the TI-89 and VOYAGE 200 graphing handhelds as well as create a user-defined program amenable to further partake of this sequence. In addition, I will furnish lists to provide the number of steps for the Collatz sequence to converge to one for the first thirty-five positive integers. A related activity will provide the initial integer which requires from one up to thirty-five steps to reach one together with observing the maximum height obtained for the first thirty-five values. A culminating exploration entails the idea of the Collatz k -tuple which consists of k consecutive integers for which the Collatz sequence has the same length. For example, $\{65, 66, 67\}$ represents a Collatz 3-tuple of length 27 (not counting the seed values). We are asserting that each of these three consecutive integers requires 27 steps to reach one. In addition, the first integer requiring more than one hundred steps to reach one is 27 needing 111 steps. The use of CAS technology as manifested by CAS graphing calculators as well as software packages such as MATHEMATICA enables the student to obtain insights independently of the teacher. The ability to partake in mathematical discovery has shaped the discipline throughout the ages which in no small measure is the beauty and the power of mathematics.

We initiate our discussion with an illustration. Let us employ the VOYAGE 200 in determining the number of iterations (27) required for the sequence $\{65, 66, 67\}$ to reach 1. Our inputs and outputs are provided in **FIGURES 1-9** below:

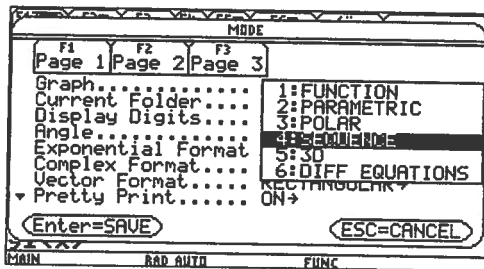


FIGURE 1: The Sequence Mode

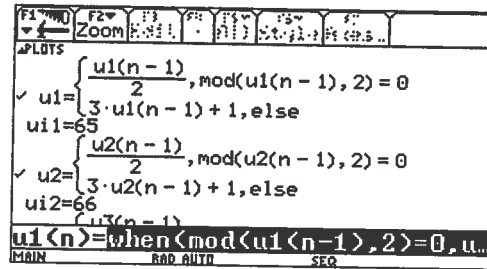


FIGURE 2: The Sequence Inputs.

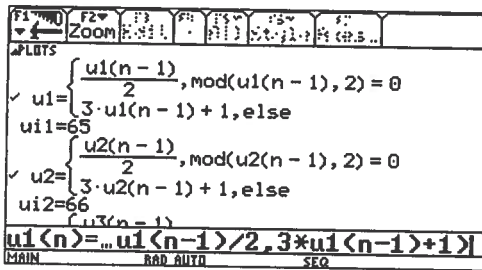


FIGURE 3: The Sequence Inputs.

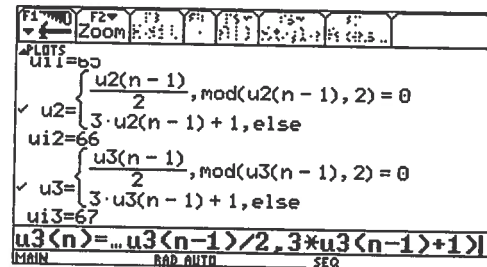


FIGURE 4: The Sequence Inputs.

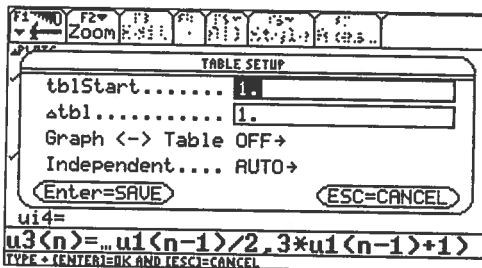


FIGURE 5: The Table Setup.

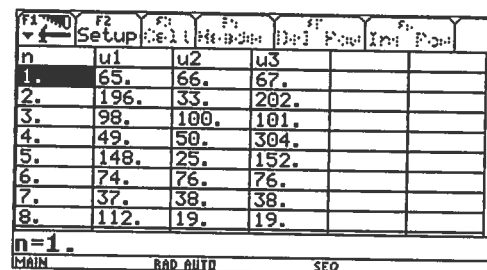


FIGURE 6: The Table Revealed.

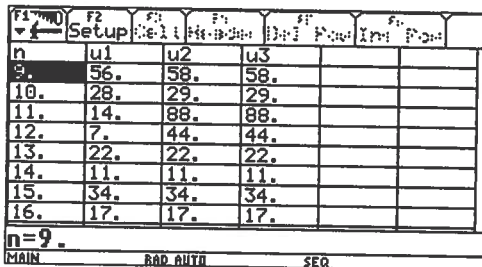


FIGURE 7: The Table Revealed.

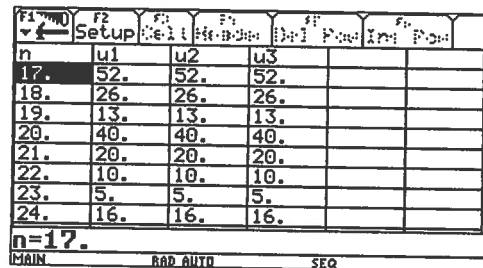


FIGURE 8: The Table Revealed.

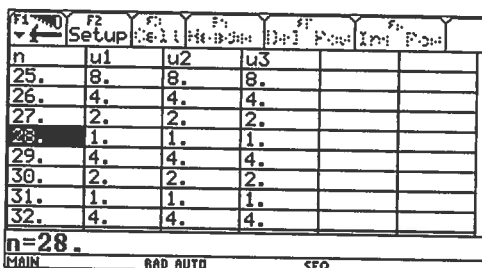


FIGURE 9: The Table Revealed.

Our initial goal is to provide a line plot for the initial thirty-five values using the TI-84+ SE graphics calculator where we form two lists L_1 and L_2 . In list L_1 we generate each of the first thirty-five counting integers and in list L_2 the number of steps required for the integer to converge to 1 obtained via MATHEMATICA. See FIGURES 10-14 below:

L1	L2	L3	2
1	8	-----	
2	1		
3	2		
4	3		
5	4		
6	5		
7	6		
8	16		

L2(1)=3

FIGURE 10: The Lists Revealed.

L1	L2	L3	2
8	8		
9	19		
10	6		
11	14		
12	9		
13	9		
14	17		

L2(8)=3

FIGURE 11: The Lists Revealed.

L1	L2	L3	2
15	4		
16	12		
17	20		
18	20		
19	7		
20	7		
21	7		

L2(15)=17

FIGURE 12: The Lists Revealed.

L1	L2	L3	2
22	15		
23	10		
24	23		
25	10		
26	111		
27	18		
28	18		

L2(22)=15

FIGURE 13: The Lists Revealed.

L1	L2	L3	2
29	18		
30	106		
31	5		
32	26		
33	13		
34	13		
35	13		

L2(29)=18

FIGURE 14: The Lists Revealed.

STAT PLOTS	
1:Plot1...On	
2:Plot2...Off	
3:Plot3...Off	
4:PlotsOff	

FIGURE 15: The Stat Plot Setup.

Plot1	Plot2	Plot3
Off	Off	Off
Type:		
Xlist:	L1	
Ylist:	L2	
Mark:		

FIGURE 16: The Stat Plot Type:

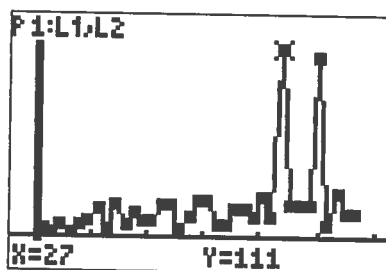


FIGURE 17: The Line Plot.

Observe in FIGURE 17 above that the coordinates $X = 27$, $Y = 111$ indicates that it takes 111 steps for the integer 27 to reach 1 using the iterative rules in this sequence.

As one tracks an integer in the Collatz sequence iterations, they notice that, in general, one has rising and falling patterns until the sequence converges to one for that integer.

Such a sequence is often called a *Hailstone sequence*. This behavior was apparent with the integers 65-67 in FIGURES 6-9 above where we observed that it took 27 steps for each of these integers to converge to 1. The pursuit of the maximum height obtained for each of the first thirty-five integers in the sequence obtained via MATHEMATICA is now illustrated in the construction of the line plot utilizing FIGURES 18-23 below:

L1	L2	L3	Z
1	7	-----	
2	16		
3	4		
4	16		
5	16		
6	52		
L2(1)=4			

FIGURE 18: The Lists Revealed.

L1	L2	L3	Z
8	8		
9	52		
10	16		
11	52		
12	16		
13	40		
14	52		
L2(8) = 8			

FIGURE 19: The Lists Revealed.

L1	L2	L3	Z
15	160		
16	16		
17	52		
18	52		
19	88		
20	20		
21	64		
L2(15) = 160			

FIGURE 20: The Lists Revealed.

L1	L2	L3	Z
22	52		
23	160		
24	24		
25	88		
26	40		
27	9232		
28	52		
L2(22) = 52			

FIGURE 21: The Lists Revealed.

L1	L2	L3	Z
29	88		
30	160		
31	9232		
32	32		
33	100		
34	52		
35	160		
L2(29) = 88			

FIGURE 22: The Lists Revealed.

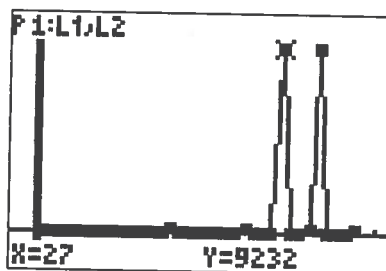


FIGURE 23: The Line Plot.

Observe in FIGURE 23 above that the coordinates $X = 27$, $Y = 9232$ indicates that the maximum height achieved for the integer 27 is 9232 using the iterative rules in the Collatz sequence. In fact, it can be shown via MATHEMATICA that 354 of the first one thousand integers reach this height.

If one desires a simple user-defined function program for the Collatz sequence on the TI-89/VOYAGE 200, we can do the following:

Proceed to SEQUENCE MODE. In the $Y =$ EDITOR, input the following function:

$$u1 = \text{when}(\text{mod}(u1(n-1), 2) = 0, u1(n-1)/2, 3 \cdot u1(n-1) + 1)$$

$$u1l = \{\text{any positive integer of your choosing}\}$$

Alternatively, on the HOME SCREEN:

$$\text{Define collatz}(n) = \text{when}(\text{mod}(n, 2) = 0, n/2, 3 \cdot n + 1)$$

Select a value for n . To perform the iterations, use `collatz (ans (1))` and press ENTER continuously until you eventually reach 1.

To cite an illustration, we determine the number of iterations needed to reach 1 for the positive integer 75. See **FIGURES 24-25** below:

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
collatz(75)					226
collatz(226)					113
collatz(113)					340
collatz(340)					170
collatz(170)					85
collatz(85)					256
collatz(256)					128
collatz(ans(1))					
MAIN	RAD AUTO			FUNC 2/99	

FIGURE 24: Iterations For 75.

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
collatz(256)					128
collatz(128)					64
collatz(64)					32
collatz(32)					16
collatz(16)					8
collatz(8)					4
collatz(4)					2
collatz(2)					1
collatz(ans(1))					
MAIN	RAD AUTO			FUNC 15/99	

FIGURE 25: Iterations For 75.

Hence 14 iterations are required, not including the initial seed value (75).

We conclude with the software package MATHEMATICA enabling one to generate neat results with the Collatz sequence. The following will serve to whet one's appetite:

```
coll[n_] := n/2 /; EvenQ[n]
coll[n_] := 3 n+1 /; OddQ[n]
collist[n_] := FixedPointList[coll, n, 1000, SameTest -> (#2 == 1 &)]
```

```
collist[16]
```

```
{16, 8, 4, 2, 1}
```

```
Table[{n, Length[collist[n]]-1}, {n, 1, 35}]
```

```
{{1, 3}, {2, 1}, {3, 7}, {4, 2}, {5, 5}, {6, 8}, {7, 16}, {8, 3}, {9, 19}, {10, 6}, {11, 14}, {12, 9}, {13, 9}, {14, 17}, {15, 17}, {16, 4}, {17, 12}, {18, 20}, {19, 20}, {20, 7}, {21, 7}, {22, 15}, {23, 15}, {24, 10}, {25, 23}, {26, 10}, {27, 111}, {28, 18}, {29, 18}, {30, 18}, {31, 106}, {32, 5}, {33, 26}, {34, 13}, {35, 13}}
```

```
Table[{n, Max[collist[n]]}, {n, 1, 35}]
```

```
{{1, 4}, {2, 2}, {3, 16}, {4, 4}, {5, 16}, {6, 16}, {7, 52}, {8, 8}, {9, 52}, {10, 16}, {11, 52}, {12, 16}, {13, 40}, {14, 52}, {15, 160}, {16, 16}, {17, 52}, {18, 52}, {19, 88}, {20, 20}, {21, 64}, {22, 52}, {23, 160}, {24, 24}, {25, 88}, {26, 40}, {27, 9232}, {28, 52}, {29, 88}, {30, 160}, {31, 9232}, {32, 32}, {33, 100}, {34, 52}, {35, 160}}
```

The initial integers requiring from 1-25 steps inclusive to reach one are respectively:
2, 4, 1, 16, 5, 10, 3, 6, 12, 24, 48, 17, 34, 11, 22, 7, 14, 28, 9, 18, 36, 72, 25, 49, 98, 33, 65, 130, 43, 86, 172, 57, 114, 39, 78.