THE BATTLE OF TRAFALGAR:
AN EXAMPLE OF MODELING WITH TECHNOLOGY

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Introduction

A classic example of the directed fire model of combat, Lanchester Square Law, is the Battle of Trafalgar. In classical naval warfare, two fleets would sail parallel to each other and fire broadside at one another until one fleet was annihilated or gave up (see Figure 1). The white fleet represents the British and the Black fleet represents the French-Spanish fleet.

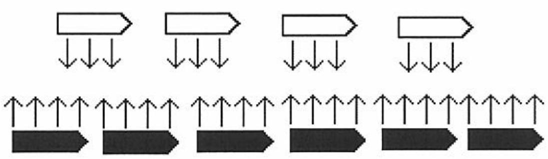

Figure 1: The White Fleet, the British, takes a beating

Lanchester's Equations as Difference Equations

In such an engagement, the fleet with superior firepower will inevitably win. To model this battle, we begin with the *system of difference equations* that models the interaction of two fleets in combat. Suppose we have two opposing forces with A_0 and B_0 ships initially, and $A(t)$ and $B(t)$ ships t units of time after the battle is engaged. Given the style of combat at the time of Trafalgar, the losses for each fleet will be proportional to the effective firepower of the opposing fleet. That is,

$\Delta A = -bB$ and $\Delta B = -aA$,

where a and b are positive constants that measure the effectiveness of the ship's cannonry and personnel and A and B are both functions of time. In preparing for the Battle at Trafalgar, Admiral Nelson assumed the coefficients of effectiveness of the two fleets were approximately equal. To keep things simple initially, we let $a = b = 0.05$. The figure and numerical listing (Table 1) below allows us to look at many different initial settings and try to ascertain a pattern in the results of the battle.

Let $A(t)$ = number of British ships remaining after period t, and
$B(t)$ = number of French-Spanish ships remaining after period t.

Using the paradigm, Future=Present + Change, we construct the model. In standard form the model would be:

$$A(n+1)=A(n)-k_1*B(n)$$
$$B(n+1)=-k_2*A(n)+B(n)$$

for kill rates k_1 and k_2.
For the battle of Trafalgar, we obtain specific equations and initial values as follows

$$A(t+1)=A(t)-0.05\ B(t),\ A(0)=27$$
$$B(t+1)=-0.05\ A(t) + B(t),\ B(0)=33$$

We could iterate these numbers to find who wins as well as prepare a graph as in figure 2:

Table 1: Numerical Solution to the battle

t	A(t)	B(t)
0	27	33
1	25.35	31.65
2	23.7675	30.3825
3	22.24838	29.19413
4	20.78867	28.08171
5	19.38458	27.04227
6	18.03247	26.07304
7	16.72882	25.17142
8	15.47025	24.33498
9	14.2535	23.56147
10	13.07542	22.84879
11	11.93298	22.19502
12	10.82323	21.59837
13	9.743315	21.05721
14	8.690455	20.57004
15	7.661952	20.13552
16	6.655176	19.75242
17	5.667555	19.41967
18	4.696572	19.13629
19	3.739757	18.90146
20	2.794685	18.71447
21	1.858961	18.57474
22	0.930224	18.48179
23	0.006135	18.43528

In this example, Admiral Nelson has 27 ships while the allied French and Spanish fleet had 33 ships. As we can see from both the table and the figure, Admiral Nelson is expected to lose all 27 of his ships while the allied fleet will lose only about 14 ships.

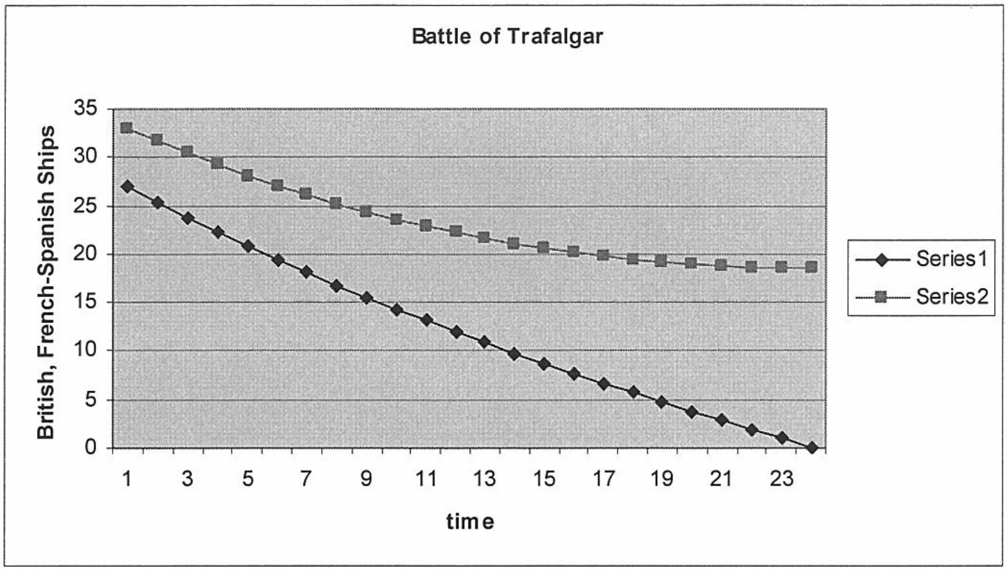

Figure 2: Battle of Trafalgar under normal battle strategies

Analytical solutions can be found using eigenvalues and eigenvectors:

$$A(k)=33\ (.95)^k - 3.0(1.05)^k$$
$$B(k)=33\ (.95)^k + 3\ (1.05)^k$$

Now, let's return to our initial equations and we can determine who wins without iterating by looking at optimal conditions, such as:
(.05)(33) > (.05)(27)
1.65 > 1.35

Since $\sqrt{k_1 k_2} \cdot FS_0 > k_1 \cdot B_0$ then the French-Spanish Fleet win. The analytical solution can be easily developed as:

$$X(k) = -3\begin{pmatrix} -1 \\ 1 \end{pmatrix}(1.05)^k + 30\begin{pmatrix} 1 \\ 1 \end{pmatrix}(.95)^k$$

In order for the British to win, we first find the values that provide then with a draw. We find the British would require 33 ships to have draw. Additionally, we find that the British would have to increase their kill effectiveness to 0.07469 to obtain a draw. Increases just beyond these values, give the British the theoretical edge. However, there were no more ships and the armaments were in place on the ships already. The only option would be a change in strategy.

We can also test this new strategy was used by Admiral Nelson at the Battle of Trafalgar. Admiral Nelson decided to move away from the course of linear battle of the day and use a "divide and conquer" strategy. Nelson decided to break his fleet into two groups of size 13 and size 14. He also divided the enemy fleet into three groups: a force of 17 ships (called B), a force of 3 ships(called A) and a force of 13 ships (called C). We can assume these as the head, middle, and tail of the enemy fleet. His plan was to take the 13 ships and attack the middle 3 ships. Then have his reserve 14 ships rejoin the attack and attack the larger force B, and then turn to attack the smaller force C. How did Nelson's strategy prevail?

Assuming all other variables remain constant other than the order of the attacks against the differing size forces, we find the Admiral Nelson and the British fleet now win the battle sinking all French-Spanish ships and 13 to 14 ships remaining.

How did we obtain these results? The easiest method was iteration and used three battle formulas. We stop each battle when one of the values gets close to zero (before going negative). We illustrate in Figure 3.

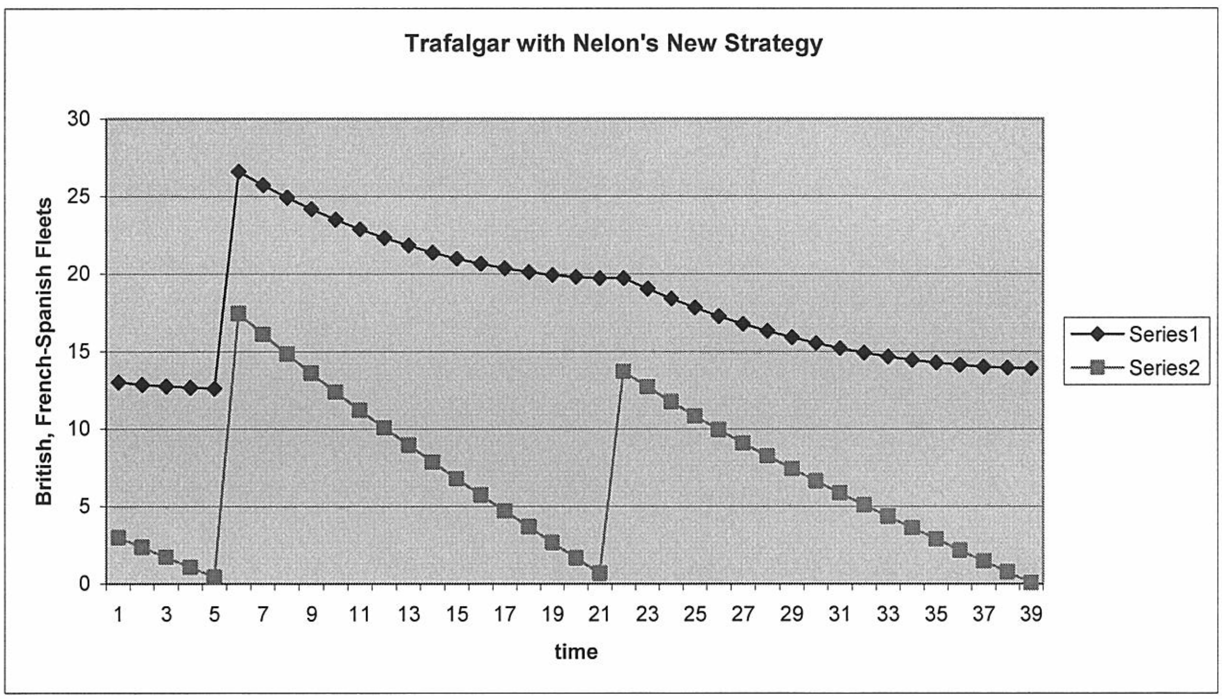

Figure 3. British prevail with new strategy.

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