SEEING IS BELIEVING — VISUALIZING CALCULUS

Adam O. Hausknecht and Robert E. Kowalczyk
University of Massachusetts Dartmouth
Mathematics Department, 285 Old Westport Road, N. Dartmouth, MA 02747-2300
rkowalczyk@umassd.edu and ahausknecht@umassd.edu

If a picture is worth 1000 words, then an animation must be worth at least 10,000 words. How can we convert this efficiency in communication to the world of teaching mathematics? In designing our software package, TEMATH, we have painstakingly designed tools that can be used to display mathematical concepts using "pictures" or animations. Additionally, we have developed a friendly, unified interface that makes it easy to use these visualization tools when presenting new mathematical topics in a calculus class. We feel that when an instructor uses technology in a pedagogical manner, it is important to make the technology as invisible as possible so that students can concentrate on the mathematics being taught. The TEMATH interface (see Figure 1) contains four panels: Graph, Work, Report, and Domain & Range. These panels are visible at all times to provide an efficient working environment. All work (function models, data sets, vector fields, …) is saved in the Work panel, all the results of mathematical computations are

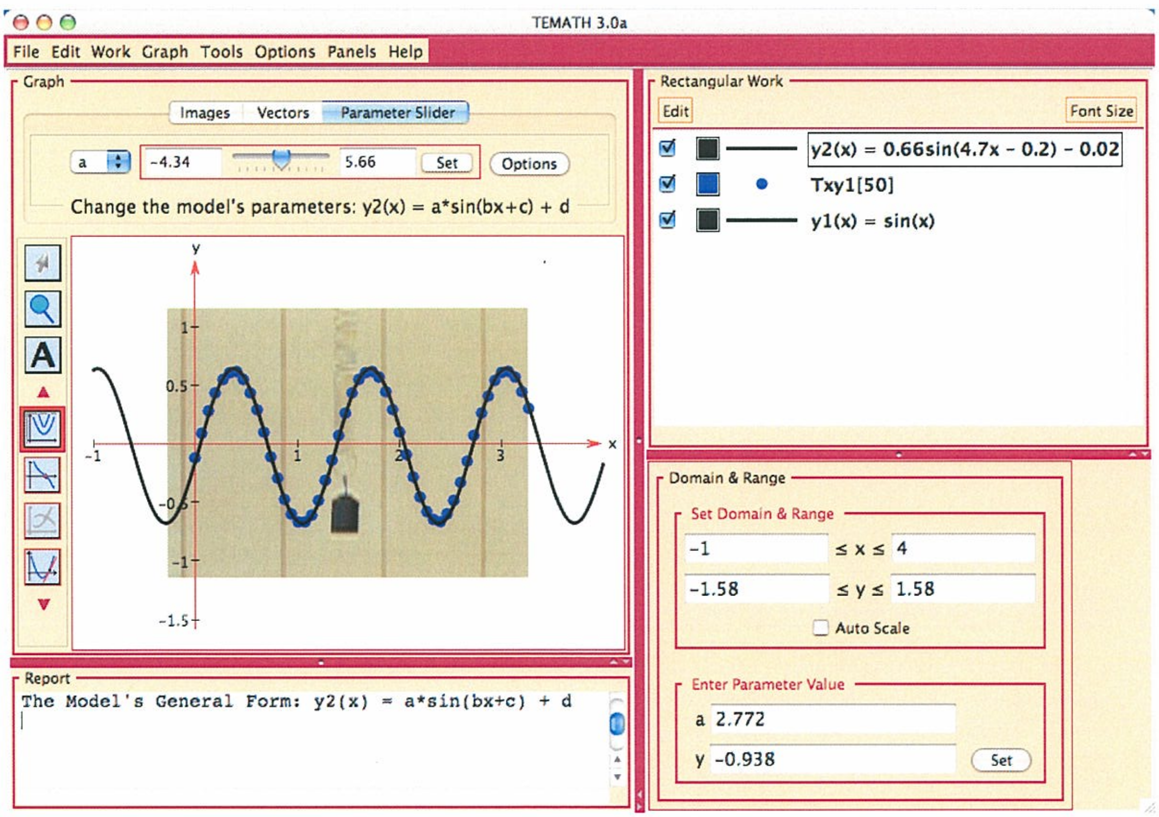

Figure 1 The TEMATH Interface

recorded in the Report panel, and all the visualizations take place in the Graph panel. The Tool palette in the Graph panel contains a robust set of visualization tools for exploring concepts in

Pre-calculus:	Parameter variation in functions, identifying functions, root finding,
Calculus I-II:	Limits, continuity, differentiation, iteration, integration, arc length,
Vector-Calculus:	Vector operations, flow lines through a vector field, line integrals,
Geometry:	Parametric curves, polar curves, curvature,
Modeling:	Fitting real-world data, and solving applied problems.

The Control panel at the top of the Graph panel contains the controls needed by the visualizations tools. We have taken many of the standard calculus animations and some not so standard and placed them in a uniform interface for easy demonstrations by teachers and explorations by students. Visualization should make classroom presentations more interesting, enlightening, and motivating. The most rewarding response from students is when they exclaim, "I see it!"

A possible scenario for integrating TEMATH's visual tools into the teaching of calculus is described below.

During the pre-calculus review at the beginning of a calculus course, you can use the Parameter Variation tool to explore the amplitude, period, horizontal shift, and vertical shift of the sine function, and then use this insight to model, say, the height of a mass oscillating on a spring set up in the front of the classroom for student viewing. Using the template $y(x) = a\sin(bx + c) + d$ for the sine function with the four parameters a, b, c, and d, the Parameter tool allows the user to change the value of each parameter and watch the effect it has on the curve plotted in the Graph panel. This allows students to associate the scaling and translating of functions with the different parameters. Once students have developed an understanding of the function of each of the parameters, they can use their new knowledge in an applied setting. For example, you can have your students model the motion of the mass-spring oscillating system. To do this using TEMATH, you need to take a short digital video clip of the oscillating spring-mass system, convert the video to a sequence of frames, and import these frames into TEMATH (this can easily be done in five minutes of time). Within TEMATH, you can step through each frame of the video and use the Point tool to mark the height of the mass. When you have marked all the frames, you can convert the points to a data set and using the Data Table Window enter the times of each frame into the first column of data (you need to know the speed of your camera, for example, 30 frames per second). Plot these points and then have your students change the values of the parameters of the sine function to find the best model for the height of the oscillating mass. When students change the value of a parameter, they get instant visual feedback on its effect on the potted curve. This is a fun exercise and students really enjoy it (see Figure 1). The Parameter tool also makes it easy to visually explore the scaling, translation, and reflection of the standard functions studied in a calculus course (trigonometric, exponential, logarithmic, polynomial, rational, hyperbolic, and logistic) by animating the changing shape of a curve as a parameter in its functional expression varies.

As your course progresses to the definition of the derivative, the Tangent tool can be used to provide students with an intuition of the concept of the derivative by seeing and investigating all aspects of the derivative. The forward and backward secant line animations provide a visual approach to developing the concept of the derivative at a point. Observing the limiting behavior of the secant lines from both directions provides students with a solid understanding of the limit process (see Figure 2). Once the definition of the derivative at a point has been developed, the Dynamic Tangent tool can be used to set the derivative function in motion. Watching the tangent move along a curve and generate the graph of the derivative function truly increases a student's understanding of the derivative as a function. This visualization provides an environment for students to conjecture or to guess what standard function best models the example derivative drawn by the Dynamic Tangent tool.

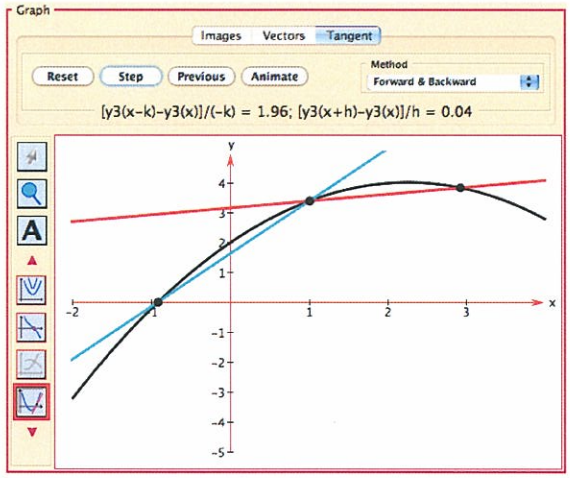

Figure 2 Forward - Backward Differences

TEMATH also contains some specialty tools for investigating topics such as root finding, iteration, and special functions. The Root Finding tool provides the necessary controls to step through or animate the convergence process for the bisection, secant or Newton method. Students can use these tools to visually explore the conditions for convergence or the problematic situations where these methods do not converge (for example, wandering, cycling). The Iteration tool generates the typical cobweb displays for finding fixed-points, orbits, and chaos, but it can also be used to study the process of generating a sequence of iterates and discussing the convergence or divergence of the sequence. Special functions, such as the hyperbolic functions, can be used in applied problems. For example, you can import a digital image of two students holding a hanging cable (chain, rope) into TEMATH as a background image and have your students find the hyperbolic cosine model that perfectly fits the image of the hanging cable.

When introducing the topic of integration, you can use TEMATH's Integration tool to display an animation of the convergence of the cumulative area of rectangles to the area under a curve. Animations are available for left-endpoints, right-endpoints, midpoints, random points, trapezoids and Simpson's rule. Convergence rates and efficiency can be compared among the approximation models. The Dynamic Integration tool provides an animation of the "area so far" function that is useful for exploring the generation of special functions such as the Sine integral $\mathrm{Si}(x)$ and the natural logarithmic function $\ln(x)$.

In the second semester of a calculus course, TEMATH's Arc Length tool can be used to animate the visual convergence of the cumulative length of line segments to the length of a curve. Watching this visualization for a variety of functions will convince your students that this theory really works and that using the limit of a Riemann sum of lengths of line segments does solve the arc length problem. Additionally, the Parametric and Polar Tracker tools can be used to help students visualize the construction of parametric and polar curves. These tracker tools have options for displaying tangent lines, normal lines, and velocity and acceleration vectors. The parametric tracker provides the student with a visualization of how a parametric curve is constructed as a function of the value of the parameter. Using this tool, it is easy to visually compare different parameterizations of the same curve. In fact, you can even change this into a game. For example, you can construct a parametric raceway by comparing the following sets of parametric equations:

$$x(t) = 4\cos(t),\ y(t) = 2\cos(t) \text{ and } x(t) = 4\cos(2t),\ y(t) = 2\cos(2t),$$

where $0 \le t \le 2\pi$. Using the Parametric Tracker, you can trace both of these curves (the same ellipse) simultaneously and watch the race proceed as one driver moves twice as fast on one ellipse than the other. You can ask students questions such as "How much of a lead do you have to give the first driver so that the race ends in a tie?"

TEMATH contains a number of visual tools for exploring topics in multivariable calculus. The Parametric Tracker tool can be used to move an osculating circle along the path of a curve to help students better understand the concept of curvature. TEMATH also contains a set of predefined vector fields that can be plotted. The Flow Line tool allows the user to click in the vector field at a point where you want a flow line to be drawn. Once the flow line is drawn, you can click on it and drag it to observe how the flow line changes as the initial point is moved around the vector field. In addition to drawing flow lines, this tool also draws contour lines. Using this tool, students build an intuition about the properties of a vector field. The Line Integral tool provides an interface for studying

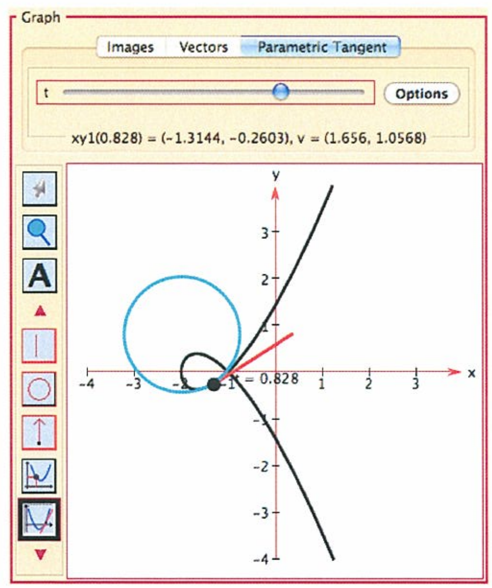

Figure 3 Osculating Circle

the properties and structure of line integrals along a parametric path through a vector field. Click the Animate button to watch the line integral progress as you move along the path. At each point along the path, the direction vector and vector field vector are displayed, and the contribution to the value of the line integral for each step is calculated and added to the value of the integral. By observing these two vectors, it is possible to visualize whether the contribution to the integral is positive or negative. Using this tool, the properties of independence of path and conservative vector fields are easily visualized and understood by students. These tools can also be used to gain insight into applied problems, such as, using Green's theorem to find the size of an enclosed area[1].

TEMATH's Vector interface allows a user to enter a vector using the click and drag method or, more exactly, by entering the coordinates of the endpoints. Additionally, we've implemented the following vector tools: scalar multiple, norm (length), angle, <i, j> components, unit vector, sum of vectors, difference of vectors, dot product, cross-product (2D), angle between vectors, linear combination of vectors, projection of one vector onto another vector, and orthogonal component of one vector with respect to another vector. To help students visualize these vector operations, we have included some animations. For example, when two vectors are added, the second vector is slowly moved to align its tail with the head of the first vector, and then the sum vector is displayed. Animations are available for the sum, difference, projection, and linear combination of vectors.

Our software is still in development. We plan to add many more visual tools for exploring and investigating other topics in calculus. Our goal is to have a common, uniform interface in which all these animations, mathematical experiments, explorations in applied problem solving, and visualizations can take place. Our interface design attempts to have the tools and information necessary for accomplishing a task readily available on the screen without searching for hidden functionality. We have an alpha version of our new cross-platform version of TEMATH available for testing. It runs on computers with Macintosh OS X, Linux, or current versions of Microsoft Windows/NT/XP/Vista operating systems that have a Java JVM 1.6 (or later installed). If you would like a copy of this version of TEMATH V3.0a, send us an email at rkowalczyk@umassd.edu or ahausknecht@umassd.edu and we'll send you a copy. We would gratefully appreciate any feedback on this version. Please report to us any bugs in the software and any suggestions for improving the interface or for adding new features. If you decide to use our software with your students, we would appreciate you sharing your experience with us.

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