

# USING MAPLE IN ADVANCED CALCULUS AND MODERN ALGEBRA

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## Introduction

Maple can be used in many ways to create *MicroWorlds* for students to explore. A *MicroWorld* is the same as a “sandbox” in programming. A *MicroWorld* defines operations that a student can perform in a structured environment. The approach has significant advantages.

1. Possible operations are restricted to the desired task and can be made context sensitive.
2. Learning advanced syntax is eliminated and can actually be hidden.

*MicroWorlds* can be created using a number of different Maple constructs. A primitive form is to provide students with a Maple worksheet with output deleted. A better method is to give students a Maple document that contains automatic initialization code. A very useful technique is to build Maplets, a Java front-end to Maple that can present windows containing graphs, input/output boxes, and several other interface elements. See “Maplets for Calculus,” an extensive set of award-winning Maplets for elementary calculus students created by Doug Meade and Phil Yasskin.

The technique we will focus on today is using embedded components in Maple. Embedded components can be buttons, combo boxes, check boxes, radio buttons, labels, text areas, sliders, various gauges, graphs, math/text displays, or code blocks. Most components have actions that can be executed via a trigger. For example, changing the position of a slider changes the value of its associated variable. When the specific trigger action occurs, a set of Maple statements can be executed. The Maple code can perform calculations and update any active embedded components in the document. Changes to components are handled with the DocumentTools package. We will center our attention on two junior level courses Modern Algebra and Advanced Calculus for our examples.

## Modern Algebra

Many Modern Algebra courses begin by extending familiar structures such as the integers. An early topic is the Euclidean algorithm. Students are familiar with the mechanics of the

division algorithm, but are likely to have never formally considered it. We wish them to make connections between these algorithms, linear combinations, and the greatest common divisor. Numerical experiments can help students build intuition.

## The Euclidean Algorithm

We present two views: the tabular form of the Euclidean Algorithm and a list of the smallest coefficients for linear combinations of the two numbers that gives the GCD. In Figure 1 we see a “code block” icon at the upper left followed by the label `Initialize Euclidean Algorithm`. A code block is a Maple 12 construct that contains a block of Maple code which is executed whenever the icon is clicked. Here it is used to define the procedures, set up the arrays, and assign the variables needed by the associated components.

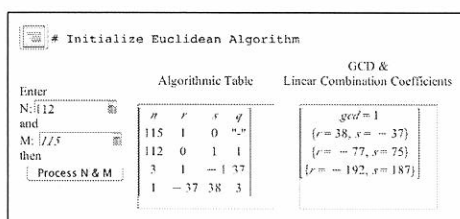


Figure 1: Euclidean Algorithm Maple Components

## Cayley Tables for $\mathbb{Z}_n$

Operation tables provide a natural lede to developing the definition of a group. We start students’ investigations with  $\mathbb{Z}_n$ , hoping to lay a basis for more abstract structures. Having students manually produce Cayley tables for  $\mathbb{Z}_n$  is time consuming and prone to error. Maple is quicker and accurate. As a bonus, we can instantly produce different size tables in class.

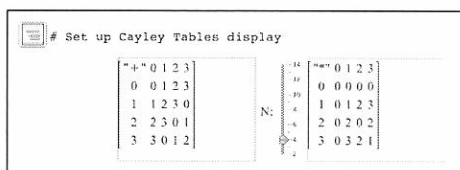


Figure 2: Cayley Tables Maple Components

## Galois Fields

At the end of an introductory course, there is often time for advanced topics to whet appetites for further study. One of my favorite topics is a low-level introduction to Galois fields. This subject gives me an opportunity to show mathematics that is being used daily as Galois fields appear in cryptography. The *Advanced Encryption System* (AES) of the US is based on the Galois field of  $GF(2^8)$  elements. For the algebraists, AES uses  $\mathbb{Z}_2[x]/\langle x^8 + x^4 + x^3 + x + 1 \rangle$ .

## Galois Field Operation Tables

While the full operation tables are too large to be useful, small segments show interesting properties. Figure 3 shows a variable size segment of  $GF(2^8)$ .

```
# Define GFadd, GFmult, and GF
a := [ [0, 1, 2, 3],
       [0, 0, 1, 2],
       [1, 1, 0, 3],
       [2, 2, 3, 0],
       [3, 3, 2, 1] ];
m := [ [0, 0, 0, 0],
       [1, 0, 1, 2],
       [2, 0, 2, 4],
       [3, 0, 3, 6],
       [0, 1, 2, 3] ];
i := [ 0, 1, 2, 3 ];
Set A;
```

Figure 3: Galois Field Maple Components

## $GF(2^8)$ Calculator

Since our tables are too small to be useful for calculations, we embed a small calculator that shows the representation of an integer and the results of the three standard arithmetic operations. The calculator lets us experiment with different values, representations, and arithmetic in the Galois field. The tie to binary is easy for students to uncover.

Enter integers  $a$  and  $m$  in  $0..255$   
 $a$ : 11 and  $m$ : 11  
 Calculate

$GF(256)$  representation:  
 $a$ :  $\{z^2 + z + 1, \text{mod} 2\}$   
 $m$ :  $\{z^2 + z + 1, \text{mod} 2\}$

$GF(256)$  arithmetic results:  
 $a + m$ : 0  
 $a \cdot m$ : 69  
 $a^m$ : 233

Figure 4:  $GF(256)$  Calculator Maple Components

## Advanced Calculus

Typically, students find the rigour of  $\epsilon$ - $\delta$  arguments to be extremely mysterious. Graphically relating the logic to convergence often helps students to build intuition.

## Sequential Limits

In Maple, define a sequence  $a_n$  as a function  $a(n)$ . We'll use  $a(n) := \sin(n^2)/n$ .

## Graphical Tool

A dynamic plot is much more effective at illustrating the arguments than a drawing. (See Figure 5.) Adjust  $\epsilon$  by moving the meter's needle. Then adjust  $N$  with the slider. The

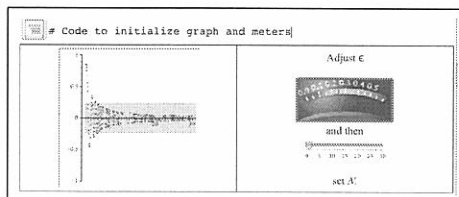


Figure 5: Sequence Graphing Maple Components

“ $\epsilon$ -strip” changes from red to blue when an appropriate  $N$  is chosen.

## A Proof in a Maple Document

Now we prove that  $\lim_{n \rightarrow \infty} |\sin(n^2)/n| = 0$ . (The following is a  $\text{\LaTeX}$  export from Maple.)

Proof. Let  $\epsilon > 0$ . Consider the inequality

$$|a(n)| < \epsilon$$

$$\left| \frac{\sin(n^2)}{n} \right| < \epsilon$$

Use the fact that sin is bounded by 1 so

`subs (sin (n^2) = 1, %)`

$$|n^{-1}| < \epsilon$$

`simplify(abs(1/n) < epsilon) assuming n :: posint`

$$n^{-1} < \epsilon$$

Choosing  $N > 1/\epsilon$  assures that  $|a_n - 0| < \epsilon$ , and the result is seen to hold.

## Function Limits

We will consider the function  $f(x) = x^3 - x$  to demonstrate investigating the limit. While this function is simple, the calculations can get in the way for student learning  $\epsilon$ - $\delta$  proofs.

### Graphical Tool

In the same fashion as with sequential limits, a dynamic graphical display helps our students to develop intuition into the process. This intuition then, in turn, helps them to understand and to write  $\epsilon$ - $\delta$  proofs. Figure 6 shows that we use a vertical slider for  $\epsilon$  this time and a text

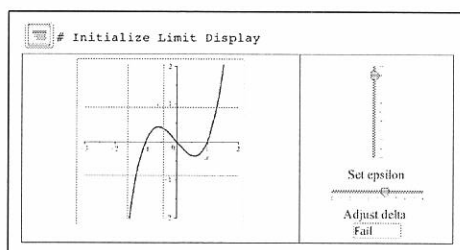


Figure 6: Function Limit Graphing Maple Components

box beneath the  $\delta$  slider to indicate an appropriate choice by changing to `Success`.

## A Proof in a Maple Document

We propose the value 0 for the limit. (The following is a  $\text{\LaTeX}$  export from Maple.)

`L := 0 :`

Consider

$$|f(x) - L|$$

$$|x^3 - x|$$

Factor to see

$$\text{factor}(|x^3 - x|)$$

$$|x(x-1)(x+1)|$$

If  $\delta < 1$ , then  $|x - 1| < \delta$  implies that  $0 < x < 2$ , that  $-1 < x - 1 < 1$ , and that  $1 < x + 1 < 3$ . Hence  $0 < x(x-1)(x+1) < 6\delta$ . Thus  $|f(x) - L| < 6\delta$ . So choose  $\delta < \min(1, 1/6\epsilon)$  and the limit  $L = 0$  is verified.

## Riemann-Stieltjes Integrals

Another of my favorite advanced topics to end an introductory advanced calculus course with is Riemann-Stieltjes integration. This integral introduces students to 20th century analysis. Students begin to see analysis as developing, not a subject that began and ended with Newton and Leibniz. We are investigating measures that have derivatives.

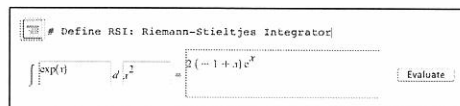


Figure 7: Riemann-Stieltjes Integrator Components

## Both Courses

Often, Mathematical Induction appears in both Modern Algebra and Advanced Calculus. Most instructors agree the duplication is good for students and enjoy presenting the topic.

## Mathematical Induction

Many students see induction proofs for the first time in Modern Algebra or Advanced Calculus. Some students get lost in arithmetic, instead of concentrating on the method, others try to be algorithmic, rather than thinking. Using Maple let's us off-load computation and focus on the concept. Another benefit is that in Maple we can define a proposition of  $n$  as an equation. (See the Maple document for details.)

## Conclusion

Embedded components can be used to build MicroWorlds for student investigations. Actions attached to embedded objects remove the need for students to learn complex syntaxes and allow them to focus on the concepts while also providing a richer environment for study and experimentation. Students can build a stronger understanding and develop deeper intuition using this facility of Maple.

For an online video tutorial on embedded components from Maplesoft, visit:

<http://www.maplesoft.com/support/training/videos.aspx>

Maple 11 and 12 worksheets for this document are available at:

[www.mathsci.appstate.edu/~wmc/ICTCM/ICTCM20/](http://www.mathsci.appstate.edu/~wmc/ICTCM/ICTCM20/)