REAL ANALYSIS ONLINE
A COMMUNITY OF LEARNERS AND THE CONCEPT OF PROOF

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Abstract
Online courses have become the medium of choice for offering courses to students who could otherwise not attend the class in a traditional classroom setting. While the internet cannot replace a traditional face-to-face interaction, research indicates that nevertheless a community of learners can be created through asynchronous discussion forums. The purpose of this article is to show how such a community can be created in the context of an upper-division mathematics class (Real Analysis), and how technology and peer interaction can be used to teach students how to write proofs.

1. Introduction

Most professionals would agree that learning is a social experience. An effective teacher actively engages students in the classroom and fosters an atmosphere of mutual support. In the context of distance education courses the necessity for building a community of learners is widely recognized in the literature (e.g., Kirschner et al., 2004; Rourke, 2001; Northrup 2000; Weegerif, 1998). The consensus of this research is that the creation of a class community in online courses is desirable because it is a prerequisite for the social aspect of learning. The present paper discusses the how asynchronous class discussion and technology in an online Real Analysis class can be used to meet the objective of this course.

The primary objective of the course is for students to prove statements in the context of real analysis. Regardless whether this course is offered as a traditional class, or in a distance education setting, students' most common trouble is producing a mathematically correct proof. These difficulties have been studied extensively in the literature (e.g. Alcock & Simpson 2004; Almeida 2000; Weber 2002). Typical problems that students encounter are: Providing a diagram instead of a definition or a proof; giving an example in order to establish a universal statement; confusing antecedent and consequent of an implication; or not knowing what to do altogether.

Examples and diagrams are an essential part of mathematical knowledge. No course would be complete without providing students with these important tools. A wide array of examples allows mathematicians to quickly generate counterexamples to incorrect statements. Diagrams provide the basis for mental models - intuitive pictoral representations of abstract objects. Fischbein (1982) and Thurston (1994) argue that mental models are a necessary precursor to formal mathematical thought and proof. The following sections will describe how the course material provided students with the tools

to understand Real Analysis, how an online discussion forum was be used to create a community of learners, and how both helped teach the concept of proof.

2. Course Concept

In a traditional class setting, students' notes are often their primary reference. In a distance education course the textbook will likely take this role. Therefore, it is important to choose a textbook that students will be able to read and understand. For this reason, the instructor of this course chose Stephen Lay's *Analysis with an Introduction to Proof* (Lay 2005). It was felt that the text would be perceived as non-threatening to students, and provide ample examples and guidance in constructing proofs.

In order to supplement the textbook, the instructor created PowerPoint presentations, which were mailed to students before the beginning of the course. These presentations follow the textbook closely, but provide narrative, animated diagrams and additional examples. Each page within the PowerPoint presentations begins with a blank screen. The instructor's voice provides explanations as text appears one sentence at a time. Diagrams within the PowerPoint are typically animated: Objects within the diagrams appear in the order in which they occur within the respective definition, thus providing a distinctive advantage over the static diagrams that would occur in a printed textbook. These diagrams provide the basis for mental models upon which students can build their intuitive understanding of the abstract concepts of Real Analysis.

The combination of Textbook and PowerPoint lectures would make this course a glorified independent study course: There would be no interaction among students and it would guarantee a very lonely learning experience. The community building-aspect of the course was primarily accomplished through the use of an asynchronous online discussion forum. Students were required to post at least one substantive question or answer during each week – a requirement that all students easily met. Most course participant posted between 100 and 200 messages in a 14 week period. We will discuss the community-building aspect of the discussion forum in section 3 below.

The majority of the homework assignments asked students to prove given statements. Homework assignments were completed in two stages: First, students wrote a rough draft, which was then reviewed by the instructor and another student. After considering the peer's and the instructor's critique, each student composed a final draft of the assignment. This process is intended to mimic the way a professional mathematician produces a proof. Section 4 below will provide more details in how this multi-stage process helped students learn how to write mathematical proofs.

Finally, the instructor spoke to all students by telephone several times in the semester. The purpose of these interviews was to provide individual guidance to students and also to verify that the student was actually completing the homework assignments him/herself.

3. Building Community

The key aspect to student engagement in any class is creating a learning environment in which students can collaborate and learn from another. The fact that students in an online class are typically geographically separated provides an additional challenge that is not present in traditional style classes. In this context, a community of learners is a group of individuals who are willing and able to help each other, are bound by a common desire to learn and have social interaction with one another (Cothrel and Williams, 1999). Defining features of a community are communication, collaboration, interaction and participation (Lock, 2002).

In this class, the community-building feature was the discussion forum. During the first week of classes, students were given the assignment to introduce themselves to other class members. This assignment was deliberately chosen so that students would begin building a community. Indeed, even during the first week of classes is became apparent that course participants were willing to help each other: Many students used this opportunity to express their concerns and hopes for the course; they offered advice to one another; and pledged to help one another with the assignments. This positive attitude prevailed throughout the semester.

The most beneficial aspect in using the discussion forum was to provide students a platform where they could test their ideas on how to prove mathematical statements. This collaborative approach allowed all students to participate in the discussion. When asked to prove statement, students often lament that they do not know what to do. By posting a question on the discussion forum, a student could consult his/her peers and get the necessary hint for getting started. An example of a typical thread of discussion is the following: One student states that he would like to express the statement in "the neighborhood of x does not contain infinitely many points of S" in formal mathematical terms, but cannot translate "infinitely many". Within hours of this student posting the question, another student offers the solution: Since the statement is "not infinitely many", one could say "finitely many", which then could be expressed as a finite set in roster notation.

In order to encourage fruitful discussion, the instructor of the course often deliberately withheld information so that students could present their own ideas and discuss the usefulness of each other's attempts. In most discussion threads students found an answer in a relatively short time (less than 24 hours of the initial post) and without additional assistance from the instructor.

The quality of the posts, and the tone used in students' online conversation clearly showed that this group of students was mutually supportive and bound by a common desire to learn. This support was even extended to those students who were not up to the task and who posted clearly elementary questions (e.g., how to solve a rational equation). Other students decided to collaborate through private email exchanges outside the discussion forum. This behavior can be compared to students in traditional classes who whisper in class or meet after class to study in the library. In summary, the asynchronous

discussion forum provided a platform in which a community of learners developed. Additional details on the aspect of community-building can be found in Pomper (2007).

4. Learning how to prove

Writing a proof is rarely a matter of simply taking a pencil to paper and filling the page with a finished version of the proof. With the exception of truly elementary statements, even professional mathematicians will not write a polished final version of a proof on a blank sheet of paper. It is therefore not conceivable that an undergraduate student would be able to write any proof without first making several drafts.

Ideally, a proof is formed by first understanding the statement. This entails analyzing its logical structure, looking up the requisite definitions, and conceiving an idea of how an argument might be formed to make the proof. Having a mental model can help make this first step. The PowerPoint lectures were designed to help students form such mental models; examples within the lectures illustrated how mental models can be used to conceive the idea for proofs. Once an idea has been conceived, mathematical discourse in the discussion forum provided students with the opportunity to test their ideas and improve on them.

For the first stage of each assignment, students were asked to hand in a draft of their proof that included an analysis of the statement, and that explicitly mentioned the mechanism by which the statement would be proved. Students were asked to identify antecedent (p) and consequent (q) of each implication, to state the logical structure of the statement (e.g. $p \rightarrow q$)and to identify any tautologies used in proving the statement (e.g., $p \rightarrow q$ is equivalent to $\sim q \rightarrow \sim p$). In addition to this analysis, students were expected to provide a detailed argument that proves the statement.

After writing a first draft of the proof, students reviewed each others' approaches. In critiquing each others' first drafts, students were asked to determine whether the argument in the peer's paper was correct, and how the proof could be made more elegant. The process of peer review provided the stronger students in the class with the opportunity to find flaws in their peers' arguments (and to argue that there was a flaw), and it provided the weaker students with a detailed approach to each proof.

After returning the peer review to the original authors, students were asked to improve on their first draft: If the argument was incorrect, they were expected to write a correct proof; if the argument was correct, they were asked to make it more elegant.

A qualitative analysis of students' papers from the beginning of the semester to the end shows an improvement in students' abilities to construct valid mathematical arguments. Likewise, a student survey suggests that students believe that the course format has improved their ability to construct and understand formal mathematical arguments.

5. Conclusion

The course concept was successful in building community among students and in teaching them how to write mathematically correct arguments.

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