GOLDBACH'S "OTHER" CONJECTURE - AN EXAMPLE OF HOW SPREADSHEETS CAN BE USED IN NUMBER THEORY

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ABSTRACT

Examples provide the impetus for much that is done in mathematical research. In this paper, the author demonstrates how spreadsheets can be used as a tool to provide such examples, and uses spreadsheets to work through a research problem posed for advanced undergraduates.

THE EXERCISE

In the text *Elementary Number Theory,* Fifth Edition, by David Burton; problem 5 of set 3.3 states: "In 1752, Goldbach submitted the following conjecture to Euler: Every odd integer can be written in the form $p+2a^2$, where p is either a prime or 1, and $a \geq 0$. Show that the integer, 5777 refutes this conjecture."

Although it seems that such a problem should have a clever, number-theoretic solution, it appears (after consulting with the author) that none exists. The way to refute this conjecture is to undertake an exhaustive consideration of all possible cases, computationally.

THE QUANDARY

Given p prime or 1, there are $\left\lfloor \sqrt{\frac{5777-p}{2}} \right\rfloor$ values of a to test. For $p=1$, there are $\left\lfloor \sqrt{\frac{5777-1}{2}} \right\rfloor = 53$ values of a to test, for $p=2$, there are $\left\lfloor \sqrt{\frac{5777-2}{2}} \right\rfloor = 53$ values of a to test, etc. All totaled, there are approximately 28,000 cases to check. Even with a calculator, this is prohibitive!

Alternatively, we can compute all values of $5777-2a^2$, for $a=1$ to 53, $(2\cdot 53^2 < 5777 < 2\cdot 54^2)$, and check to see if any of these are primes. The computations involved are again too numerous to take on without the use of a computer.

THE SOLUTION

In this case, technology, in the form of a spreadsheet, comes to the rescue. By creating a spreadsheet, and using relative addressing and the copy command, we can easily create a table of values of $5777 - 2a^2$, for $a = 1$ to 53. (Note that $2 \cdot 53^2 < 5777 < 2 \cdot 54^2$.) Then, using the copy command again, we can make a copy of this table for $p = 1$, as well as for each prime p from 3 to 59, and test each value of $5777 - 2a^2$ for divisibility by each value of p, to show that the conjecture fails. ($p = 59$ finally exhausts all possibilities.)

A copy of the table for $p = 3,5,13$ is shown in Figures 1a and 1b. FYI, the command for the cells in the column headed by "Div by 3?" is @IF(C2 - @INT(C2) = 0, "Yes", " ").

a =	5777-2a^2	(5777-2a^2)/3	Div by 3?	*	(5777-2a^2)/5	Div by 5?	*	(5777-2a^2)/13	Div by 13?
1	5775	1925	Yes	*	1155	Yes	*	444.230	
2	5769	1923	Yes	*	1153.8		*	443.769	
3	5759	1919.666		*	1151.8		*	443	Yes
4	5745	1915	Yes	*	1149	Yes	*	441.923	
5	5727	1909	Yes	*	1145.4		*	440.538	
6	5705	1901.666		*	1141	Yes	*	438.846	
7	5679	1893	Yes	*	1135.8		*	436.846	
8	5649	1883	Yes	*	1129.8		*	434.538	
9	5615	1871.666		*	1123	Yes	*	431.923	
10	5577	1859	Yes	*	1115.4		*	429	Yes
11	5535	1845	Yes	*	1107	Yes	*	425.769	
12	5489	1829.666		*	1097.8		*	422.230	
13	5439	1813	Yes	*	1087.8		*	418.384	
14	5385	1795	Yes	*	1077	Yes	*	414.230	
15	5327	1775.666		*	1065.4		*	409.769	
16	5265	1755	Yes	*	1053	Yes	*	405	Yes
17	5199	1733	Yes	*	1039.8		*	399.923	
18	5129	1709.666		*	1025.8		*	394.538	
19	5055	1685	Yes	*	1011	Yes	*	388.846	
20	4977	1659	Yes	*	995.4		*	382.846	
21	4895	1631.666		*	979	Yes	*	376.538	
22	4809	1603	Yes	*	961.8		*	369.923	
23	4719	1573	Yes	*	943.8		*	363	Yes
24	4625	1541.666		*	925	Yes	*	355.769	
25	4527	1509	Yes	*	905.4		*	348.230	
26	4425	1475	Yes	*	885	Yes	*	340.384	
27	4319	1439.666		*	863.8		*	332.230	
28	4209	1403	Yes	*	841.8		*	323.769	

Figure 1a. Table for $p = 3,5,13$

a =	5777-2a^2	(5777-2a^2)/3	Div by 3?	*	(5777-2a^2)/5	Div by 5?	*	(5777-2a^2)/13	Div by 13?
29	4095	1365	Yes	*	819	Yes	*	315	Yes
30	3977	1325.666		*	795.4		*	305.923	
31	3855	1285	Yes	*	771	Yes	*	296.538	
32	3729	1243	Yes	*	745.8		*	286.846	
33	3599	1199.666		*	719.8		*	276.846	
34	3465	1155	Yes	*	693	Yes	*	266.538	
35	3327	1109	Yes	*	665.4		*	255.923	
36	3185	1061.666		*	637	Yes	*	245	Yes
37	3039	1013	Yes	*	607.8		*	233.769	
38	2889	963	Yes	*	577.8		*	222.230	
39	2735	911.666		*	547	Yes	*	210.384	
40	2577	859	Yes	*	515.4		*	198.230	
41	2415	805	Yes	*	483	Yes	*	185.769	
42	2249	749.666		*	449.8		*	173	Yes
43	2079	693	Yes	*	415.8		*	159.923	
44	1905	635	Yes	*	381	Yes	*	146.538	
45	1727	575.666		*	345.4		*	132.846	
46	1545	515	Yes	*	309	Yes	*	118.846	
47	1359	453	Yes	*	271.8		*	104.538	
48	1169	389.666		*	233.8		*	89.923	
49	975	325	Yes	*	195	Yes	*	75	Yes
50	777	259	Yes	*	155.4		*	59.769	
51	575	191.666		*	115	Yes	*	44.230	
52	369	123	Yes	*	73.8		*	28.384	
53	159	53	Yes	*	31.8		*	12.230	

Figure 1b. Table for $p = 3,5,13$ (continued)

THE RESULTS

We discover that prior to testing $5777 - 2a^2$ for divisibility by $p = 59$, the number, $5777 - 2a^2$ with $a = 33$, is still a candidate for being prime. (At this point, the number $5777 - 2a^2$ has already been shown NOT to be prime for all other allowable values of a.) Alas, we find that $59|(5777 - 2a^2)$ when $a = 33$.

Showing Goldbach's conjecture to be false proves to be a MINOR benefit. As we can see, the spreadsheet is RICH with patterns that would, as a former colleague puts it, "send shivers up and down the spine of anyone with the slightest bit of mathematical sensitivity!" Without the use of a spreadsheet, these are patterns that would remain obscured. But now that the patterns are EXPOSED, we get to ask our students: "Why?" and "What does it mean?" For our stronger students, we can ask: "What patterns do you notice, and why do these patterns exist?" For those who are not so competitive, we can lead them through the maze, by asking the right questions in the proper sequence. For example:

QUESTIONS

1. Note the pattern that exists in the column with heading "Div by 3?" and explain WHY this pattern occurs. (Hint: Since 3 divides $5777 - 2a^2$ for $a = 1$, this is equivalent to saying that $3|5775$. Thus, $3|(5777 - 2a^2) \Leftrightarrow 5777 - 2a^2 \equiv 0 \pmod 3 \Leftrightarrow 5775 - 2(a^2 - 1) \equiv 0 \pmod 3$.)

A follow up of this hint yields:
$2(a^2 - 1) \equiv 0 \pmod 3$ (Recall that $3|5775$)
$\Leftrightarrow (a^2 - 1) \equiv 0 \pmod 3$ (Because 3 is prime)
$\Leftrightarrow (a + 1)(a - 1) \equiv 0 \pmod 3$
$\Leftrightarrow (a + 1) \equiv 0 \pmod 3$ or $(a - 1) \equiv 0 \pmod 3$ (Again, because 3 is prime)
$\Leftrightarrow a \equiv 1 \pmod 3$ or $a \equiv 2 \pmod 3$.

Having explained the pattern in the case of $p = 3$, students are better equipped to explain the patterns that occur for all other primes.

2. Note the pattern that exists in the column with heading "Div by 5?" and explain WHY this pattern occurs.

This turns out to be pretty much the same song and dance.

$5|(5777 - 2a^2) \Leftrightarrow 5777 - 2a^2 \equiv 0 \pmod 5 \Leftrightarrow 5775 - 2(a^2 - 1) \equiv 0 \pmod 5$
$\Leftrightarrow 2(a^2 - 1) \equiv 0 \pmod 5 \Leftrightarrow (a^2 - 1) \equiv 0 \pmod 5 \Leftrightarrow (a + 1)(a - 1) \equiv 0 \pmod 5$
$\Leftrightarrow a + 1 \equiv 0 \pmod 5$ or $a - 1 \equiv 0 \pmod 5 \Leftrightarrow a \equiv 1 \pmod 5$ or $a \equiv 4 \pmod 5$.

3. Based on the observations already made, what pattern do you expect to find in the column headed by "Div by 11?" and why?

Having seen the pattern in the cases of $p = 3$ and $p = 5$, the student can either guess, based strictly on the *visua*l pattern observed thus far, or they can analyze the general situation for an arbitrary prime divisor, p. Namely,

$p|(5777 - 2a^2) \Leftrightarrow 5777 - 2a^2 \equiv 0 \pmod p \Leftrightarrow 5775 - 2(a^2 - 1) \equiv 0 \pmod p$
$\Leftrightarrow 2(a^2 - 1) \equiv 0 \pmod p \Leftrightarrow (a^2 - 1) \equiv 0 \pmod p \Leftrightarrow (a + 1)(a - 1) \equiv 0 \pmod p$
$\Leftrightarrow a + 1 \equiv 0 \pmod p$ or $a - 1 \equiv 0 \pmod p \Leftrightarrow a \equiv 1 \pmod p$ or $a \equiv p - 1 \pmod p$.

For all cases, $p = 3,5,7,11$, examined thus far, this turns out to be true.

4. Make a similar prediction for the column with heading "Div by 13?"

    a. Why does this conjecture fail?

Our conjecture was based on the fact that in all previous observations, p had been a divisor of $5777 - 2a^2$, for $a = 1$, and our conjecture for the general case assumes that this is true. However, in this case, $13 \nmid (5777 - 2a^2)$ for $a = 1$.

    b. How can we revise this conjecture, and why does the revised conjecture work?

Since 13 DOES divide $5777 - 2a^2$ for SOME values of a, find the first such value on the table, and use that as a "reference." In this case, $13|(5777 - 2a^2)$ for $a = 3$. Thus, $13|5759$. This yields:

$$13|(5777 - 2a^2) \Leftrightarrow 5777 - 2a^2 \equiv 0 \pmod{13} \Leftrightarrow 5759 - 2(a^2 - 9) \equiv 0 \pmod{13}$$
$$\Leftrightarrow 2(a^2 - 9) \equiv 0 \pmod{13} \Leftrightarrow (a^2 - 9) \equiv 0 \pmod{13} \Leftrightarrow (a + 3)(a - 3) \equiv 0 \pmod{13}$$
$$\Leftrightarrow a + 3 \equiv 0 \pmod{13} \text{ or } a - 3 \equiv 0 \pmod{13} \Leftrightarrow a \equiv 3 \pmod{13} \text{ or } a \equiv 10 \pmod{13}.$$

5. How can we generalize this conjecture for an arbitrary prime, p?

Let k be the smallest positive value of a such that $p|(5777 - 2a^2)$. Then we have:

$$p|(5777 - 2a^2) \Leftrightarrow 5777 - 2a^2 \equiv 0 \pmod{p} \Leftrightarrow (5777 - 2k^2) - 2(a^2 - k^2) \equiv 0 \pmod{p}$$
$$\Leftrightarrow 2(a^2 - k^2) \equiv 0 \pmod{p} \Leftrightarrow (a^2 - k^2) \equiv 0 \pmod{p} \Leftrightarrow (a + k)(a - k) \equiv 0 \pmod{p}$$
$$\Leftrightarrow a + k \equiv 0 \pmod{p} \text{ or } a - k \equiv 0 \pmod{p} \Leftrightarrow a \equiv k \pmod{13} \text{ or } a \equiv p - k \pmod{p}.$$

6. What conclusion can we make from looking at the entries in the column with heading "Div by 17?" and why?

We can conclude that $17 \nmid (5777 - 2a^2)$ for $a = 1, 2, 3, \ldots$. Otherwise, if 17 DID divide $5777 - 2a^2$, then, based on the previous observation, ANY consecutive string of 17 values of a would produce two values of $5777 - 2a^2$ that are divisible by 17. Since the first 17 values of a produce NO values of $5777 - 2a^2$ that are divisible by 17, we must conclude that no such value of a exists.

CONCLUSION

The value of the spreadsheet, in this case, and I suspect, many other cases, is that it can be used to reveal patterns that would otherwise remain hidden. Once such patterns are exposed, they demand explanation, and in turn invite exploration.