

**MATHEMATICS LABORATORY EXPERIENCE:  
VALUE ADDED, NOT JUST AN ADD-ON**

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The natural sciences have always had a laboratory component, recognizing that one needed to tie the “real” science to the theory being taught. This hands on activity was the real heart of the subject, whether physics, chemistry, biology or whatever. Mathematics has been hampered by both the quantity of material we seek to cover and the limited time in which to do it. 2500 years ago a lifetime could have been devoted to geometry, exploring lots of different problems and from them inferring basic principles and concepts. Now an abbreviated version of geometry is merely a stepping stone to analytic geometry, which in turn is a step toward calculus, which in turn is a step toward differential equations and linear algebra and so on through the upper levels of mathematics.

The traditional resolution of this quantity coverage versus time tension has, not surprisingly, been to favor quantity coverage. A limited sample of examples on a particular topic are presented in the classroom and the textbook for the student to model their solutions to homework exercises, then you move to the next topic. This piecemeal approach to teach a wide array of solutions presents mathematics as: a problem shows up from somewhere as yet unspecified, you classify it as a type of known problem, then apply the learned technique. A solution becomes an answer you can put a box around.

This may be deemed appropriate in a service course for engineers or science majors (though I would argue that a liberal education for all students requires the same depth of understanding of principles as well as techniques for all majors), but more is needed if students are to be able to analyze and solve new problems that don’t match the learned patterns.

How does one give students the opportunity to explore a topic of consequence, build a substantial set of specific examples, draw an inference of a general principle from them and test the hypothesis, all within the time constraints of a college calculus course?

It is the computer which is making this scenario a reality, to provide a laboratory

approach to mathematics in the same way as science laboratory experiences taught chemistry and physics. In the past, time was a drawback: how to do enough special cases to have a basis for generalizing took a lot of time and effort. Even a topic such as Newton's method was taught as theory only, too tedious to actually do in practice – but now it can be done in a single lab period (with adequate theoretical preparation, of course). The mathematical laboratory can now take its rightful place among the other science laboratories, with mathematics students doing the same guided discovery experiments as other science students.

Valparaiso University has more than three decades' experience with using computers in our calculus classes. Over the years, assignments have changed dramatically as the technology available to students has increased in power and sophistication. Today Calculus I and II are taught in a 3+2 hour lecture/laboratory course, with the laboratory assignments designed to complement and extend the work done in class.

This long experience has made us aware of the challenges there are in creating a successful laboratory assignment: The assignment must be

- Something that could not be done without the computer;
- Worth doing, not just busy work – don't waste student time;
- Challenging enough to maintain interest, but doable;
- Broad enough in scope to provide opportunities for students to think;
- Focussed enough to be feasible;
- Structured to stress communication of results, not just a mass of printouts.

The overall goal is to get the students to perceive the computer as a useful and thus a beneficial and not merely an additional burden.

The lab assignment described below started as an interesting example of applications of parametric equations, with a side effect of pretty pictures to help keep student interest. In experimenting with it, I decided to include arc length of both epicycloids and hypocycloids, just to illustrate the fact that it could be done, which one wouldn't even consider if doing it without a computer. I was surprised to see arc lengths of such complicated figures came out to be not just nice but integer values (when radii of the circles were integers). This unexpected result became a focus of the assignment, with students led into discovering it for themselves, the same way I found it.

### **Parametric Equations and Arc Length**

Theoretical Preparation for the lab: The first introduction to the idea of parametric equations was done in a standard classroom setting, with motivating example: find the

equation of motion for a projectile with initial position  $(x_0, y_0)$  and initial velocity  $v_0$  at an angle of  $\theta$ . The relatively simple parametric equations

$$x(t) = v_0 \cos(\theta) t + x_0 \qquad y(t) = -\frac{1}{2}gt^2 + v_0 \sin(\theta) t + y_0$$

which can be converted into the more complicated non-parametric form

$$y = -\frac{g}{2} \left( \frac{x - x_0}{v_0 \cos(\theta)} \right)^2 + v_0 \sin(\theta) \cdot \frac{x - x_0}{v_0 \cos(\theta)} + y_0.$$

The first parametric equations lab was designed to familiarize the student with the effects of changing the constants in the parametric equations

$$x(t) = h + p \cos(at), \quad y(t) = k + q \sin(bt)$$

in a guided sequence

$$x(t) = h + \cos(t), \quad y(t) = k + \sin(t) \quad \{\text{Translation of the graph}\}$$

$$x(t) = p \cos(t), \quad y(t) = q \sin(t) \quad \{\text{Circles can become ellipses}\}$$

$$x(t) = \cos(at), \quad y(t) = \sin(bt) \quad \{\text{Lissajous figures when } a \neq b, a, b \text{ integers}\}$$

The next class built on this lab with the introduction of the equations for a cycloid, the path of a point on the rim of a wheel traveling along a straight line. This was followed by the question:

What is the path of a wheel traveling around the rim of another circle?

Why would anyone ever think of that? The author is the proud possessor of a Super Spirograph, a toy no longer available in its original form (a poor imitation with design flaws is still available). Geared wheels of varying diameter can be pinned to a board, and other wheels revolved around or inside them, meshing gear teeth helping to prevent slippage. One can thus generate pretty designs, with colored pens adding an artistic effect. Since the spirograph is no longer available, how can one create a virtual spirograph with the computer?

The Lab investigated both epicycloids (a circle of radius  $b$  rolling around the outside of a circle of radius  $a$ ) and a hypocycloid (a circle of radius  $b$  rolling around the inside of a circle of radius  $a$ ). The equations are

### Epicycloid

$$x(t) = (a+b)\cos(t) - b\cos\left(\frac{a+b}{b}t\right)$$
$$y(t) = (a+b)\sin(t) - b\sin\left(\frac{a+b}{b}t\right)$$

### Hypocycloid

$$x(t) = (a-b)\cos(t) + b\cos\left(\frac{a-b}{b}t\right)$$
$$y(t) = (a-b)\sin(t) - b\sin\left(\frac{a-b}{b}t\right)$$

Students were asked to graph epicycloids (Figure 3) and hypocycloids (Figure 4), then to describe in their own words how  $a$  and  $b$  related to the shape of the curves (number of loops/points, size), and to determine the minimum range of  $t$  values required to get a complete curve.

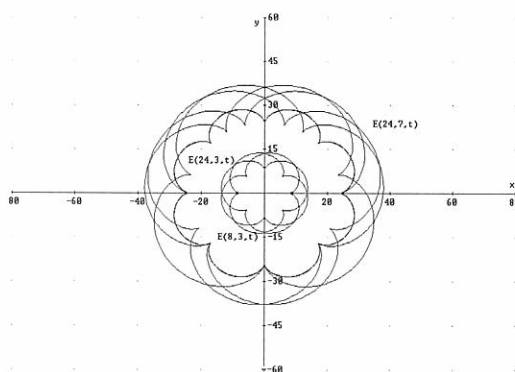


Figure 3: Epicycloids  $E(a,b,t)$

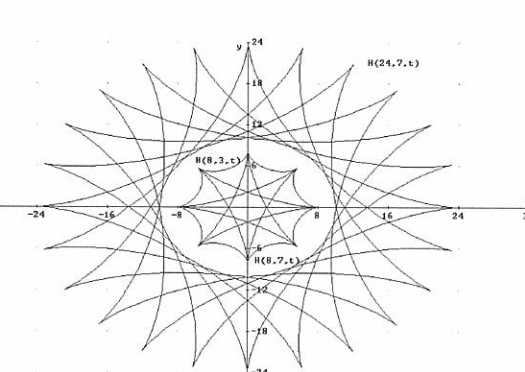


Figure 4: Hypocycloids  $H(a,b,t)$

For the epicycloid  $E(a,b,t)$  or the hypocycloid  $H(a,b,t)$ :

Reduce  $\frac{a}{b}$  to  $\frac{m}{n}$ , where  $\frac{m}{n}$  is in lowest terms.

Complete graph for  $0 \leq t \leq 2n\pi$ .

$m$  loops, outer radius  $a + 2b$  for the epicycloid,  $a$  for the hypocycloid

Interesting special case: If  $a = 2b$ , the hypocycloid is a straight line.

Students were quick to find how  $a$  and  $b$  related to the shape of the graph, but were challenged in how to put this into words. For the range of  $t$  to get a complete graph, the students' first attempts were usually just a random assortment of  $a$  and  $b$  values. Guidance was needed on how to proceed: fix one parameter, say  $a$ , then vary  $b$  until enough cases seemed to show a pattern. Change  $a$  and vary  $b$  and see if it holds up. Predict what would happen with different  $a$  and  $b$  and test the hypothesis, the scientific method in a mathematics setting.

Arc length proved more challenging. Here a consistent approach was especially important. A few students found the pattern very quickly, most found it (with a little help) by the end of the period, a few needed some very broad hints to lead them to the

correct form.

Arc length of a complete epicycloid is  $8b(a + b)$ , of a hypocycloid  $8b(a - b)$ . I and they find this a most amazing result, that if  $a$  and  $b$  have integer values, curves generated by a circle rolling on or in another circle has integer arc length, a result that can be derived explicitly (and I have done so), but only after I had inferred it from experiment on the computer.

A point I like to make is that in our teaching we avoid the hardest part of the discipline of mathematics. Given an equation or a system of equations, we teach how to find a solution or to show no solution exists. Given a word problem, we teach how to translate it into mathematical notation, find a solution and communicate it in proper form. Given an integration problem, we teach techniques of integration and strategies for deciding which technique to apply. We teach how to prove a theorem, given some definitions and the theorem to be proven. What we do not show is how to find the problem or the theorem in the first place. It is like being taught to use tools but never to create with them, to be a technician waiting for instructions what to do, in the same way as a calculator waits for its buttons to be pushed.

In this lab we have an example of how to look at a child's toy and from it to investigate – though “play with” better captures the spirit of creative mathematics – and enjoy some pretty pictures. The pictures lead to a recognition of patterns, which leads to a level of abstraction about the graphs. It is interesting to do something just because you can – in this case to have the computer approximate the arc length. You know ahead of time it is just going to lead to meaningless decimal approximations, given the complicated equations for the curves. Then there is the wonder as integers appear instead.

Much of the appeal of mathematics is the intriguing hunt for patterns in seemingly random events and situations, of surprise at simplicity where complication was expected. A proof is the end of the quest, a means to confirm an inference but not an end in itself.