UNDERDAMPED MOTION REVISITED : THREE MAPLE PROJECTS FOR DIFFERENTIAL EQUATIONS

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INTRODUCTION

I have been writing and assigning group Maple projects in my Differential Equations courses for a number of years now. I refer to these projects as "structured" in the sense that they are written in the form of a Maple worksheet which guides students through completing the project. This guidance allows for a minimal knowledge of Maple by the student since in many (although not all) cases, examples of the Maple syntax are blended into the presentation and precede the questions asked in the project. Other questions asked in these projects might, for example, produce a plot or an animation and require detailed explanations of these by the student. The idea is that students should learn some Maple, but not be overburdened by the Maple syntax since this would certainly take away from the main idea of using the technology to aid in understanding the mathematics involved.

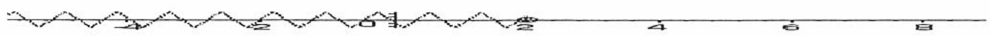
The projects I will be discussing all relate to the underdamped motion of a simple mass spring system. The disk which I will be distributing (described below) was produced in Maple 11 (the "classic worksheet" version) and represents a substantial revision and improvement of earlier work done in Maple 9.

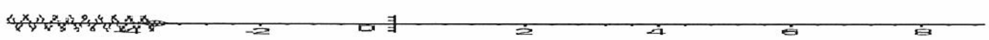
THE DISK

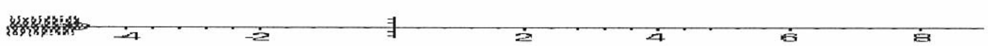
During my presentation at ICTCM20, I will be distributing a disk which contains a set of three structured Maple projects for differential equations (Underdamped Motion Projects I , II, and III), as well as complete solutions to each project. Although I retain the copyright to this material, I hope that those who receive the material will be able to use some of it in their differential equations classes. I would ask, however, that you do not provide students copies of the project solutions or publish any of these solutions to the web. Each of these projects relates the the following problem:

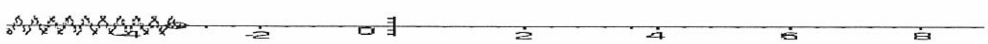
THE UNDERDAMPED MOTION PROBLEM

We consider a simple mass-spring system which is set into a back and forth motion by stretching or compressing the spring from equilibrium and perhaps giving the mass an initial velocity to the right (positive) or to the left (negative). We will assume that no outside forces act on this system--that is, the motion is "free":









If k is the spring constant and if we assume that resistance is proportional to the velocity with a constant of proportionality c (sometimes called the damping constant), then if x represents the position of the mass m at time t, we have:

$$m\left(\frac{d^2}{dt^2}x(t)\right)+c\left(\frac{d}{dt}x(t)\right)+k\,x(t)=0$$

. To simplify the form of the solution of this differential equation we will make a temporary change of variables.

Let $p=\frac{c}{2\,m}$ and $\omega_0=\sqrt{\frac{k}{m}}$. Then our differential equation can be written as:

$$\left(\frac{d^2}{dt^2}x(t)\right)+2\,p\left(\frac{d}{dt}x(t)\right)+\omega_0^{\;2}\,x(t)=0$$

. This equation is second-order linear and homogeneous. The auxiliary equation is:

$r^2+2\cdot p\cdot r+\omega_0^2=0.$

We can use Maple to help us find the solution of this differential equation. We find that the roots of the auxiliary equation are:

$$-p+\sqrt{\left(p^2-\omega_0^2\right)}\;,\;-p-\sqrt{\left(p^2-\omega_0^2\right)}\;.$$

In this project we will consider *only* the case where $p^2-\omega_0^{\;2}<0$. Observe that since:

$p^2-\omega_0^{\;2}=\frac{c^2}{4\,m^2}-\frac{k}{m}\;=\;\frac{c^2-4\,k\,m}{4\,m^2}$, this is equivalent to the case where $c^2-4\,k\,m<0$. In this case the roots are the complex conjugates $-p+i\sqrt{\omega_0^{\;2}-p^2}$ and $-p-i\sqrt{\omega_0^{\;2}-p^2}$. If we use the notation: $\omega_1=\sqrt{\omega_0^{\;2}-p^2}\;=\;\frac{\sqrt{4\,k\,m-c^2}}{2\,m}$, then the solution to the differential equation can be written in the compact form:

$x:=t\rightarrow e^{(-p\,t)}\,(A\cos(\omega_1\,t)+B\sin(\omega_1\,t))$

Observe that the Maple statement above makes the position x a function of time t. This solution represents exponentially damped motion about equilibrium and is sometimes called "damped harmonic motion". Of course A and B are constants which can be determined if we are given appropriate initial conditions. So we next determine the general solution to the underdamped problem if we have the initial position (alpha) and the initial velocity (beta). That is: x(0) = α and x ' (0) = β . Using Maple, it is not hard to show that the solution of this initial value problem is:

$$x=\;e^{\left(-\frac{c\,t}{2\,m}\right)}\left(\alpha\cos\left(\frac{\sqrt{4\,k\,m-c^2}\;t}{2\,m}\right)+\frac{2\left(\frac{c\,\alpha}{2\,m}+\beta\right)m\sin\left(\frac{\sqrt{4\,k\,m-c^2}\;t}{2\,m}\right)}{\sqrt{4\,k\,m-c^2}}\right).$$

A brief description of each of the the projects follows.

1. FREE UNDERDAMPED MOTION I: A detailed analysis of the free, underdamped motion of a mass-spring system. We consider a simple mass-spring system which is set into a back and forth motion by stretching or compressing the spring from equilibrium and perhaps giving the mass an initial velocity to the right (positive) or to the left (negative). We will assume that no outside forces act on this system--that is, the motion is "free". This project develops general formulas for the pseudoperiod, times the mass passes through equilibrium, the time between successive extrema, and the familiar "cosine" form of the solution. Some analysis of three-dimensional surfaces is required by the student.

2. FREE UNDERDAMPED MOTION II: Refer to 1 above for the basic problem of free underdamped motion of a mass-spring system. In Part I we developed formulas for the solution of the free underdamped motion of a mass on a spring and various quantities related to this motion. In this project (which is completely self-contained) we continue this investigation. However, in this project, many of the required formulas from Part I will be stated without proof or derivation. If you require more details about these formulas, you might want to refer to Part I. However, this project is completely independent of Part I. After some preliminaries and an example, our main focus in this project is to investigate the effect of holding all parameters in the solution constant except one. We will concentrate on the three cases of : (i) varying the mass m; (ii) varying the damping coefficient c; (iii) varying the spring constant k.

3. FREE UNDERDAMPED MOTION III: As in the project FREE UNDERDAMPED MOTION II, formulas developed in FREE UNDERDAMPED MOTION I will be once again stated without proof or derivation. If proofs or derivations of these formulas are required, refer to the project FREE UNDERDAMPED MOTION I above. Except for this fact, this project is completely independent of the previous ones. After some preliminaries and an example, our main focus in this project is to investigate the effect of holding all parameters in the solution constant except one. We will concentrate on the two cases of :

(i) varying the initial position α

(ii) varying the initial velocity β.

THE WEB SITE

The web site given below contains a sample of the differential equations Maple projects as well as some additional Maple worksheets. Please feel free to access this material and to use whatever you find appropriate in your own classes. The copyright on this material is, however, retained by myself.

http://www2.SPSU.edu/math/fadyn/index.html

The material can also be found by beginning at the Southern Polytechnic State University home page and then navigating to Professor Fadyn's home page. The home page for Southern Polytechnic State University is located at: http://www.spsu.edu/