

EXPLORING LIMITS WITH MAPLE

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Abstract

In this paper, we will demonstrate the use of Symbolic Computation Systems, specifically Maple, in investigating limits. We have selected examples from various areas in undergraduate mathematics and have made a modest attempt to utilize technology to visualize and explore properties of the limits and calculate or estimate some of the hard-to-determine limits.

Keywords: Maple; Symbolic Computation Systems, limits, convergence

Introduction

We have over two decades of experience in teaching mathematics and have used technology in our mathematics classes for over fifteen years. We sincerely believe that many students in undergraduate mathematics courses do not fully appreciate the fine points of limits. Some, except for trivial cases such as polynomials or rational functions, fail to appreciate the complexities of limits such as limits of transcendental functions or functions of several variables and others, have difficulty understanding the concept of limits when is used in conjunction with topics such as convergence of infinite series, Taylor series or Fourier Series and determining or at least estimating such limits. The symbolic and computational power of a Symbolic Computation System or a Computer Algebra System (CAS) allows us to design experiments, which helps students to overcome some of these challenges. We have chosen most of our examples from topics related to limits in undergraduate mathematics as they appear in courses such as calculus, differential equations and foundations of applied mathematics to make our presentation accessible to a larger audience. To show the danger of over-reliance on technology, we have also presented examples where Maple, in attempting to calculate a limit, provides inaccurate or misleading results.

We have used Maple10 as a representative of the Symbolic Computation Systems mainly because Maple appears to be the software of choice for most of the undergraduate mathematics departments in the US and Canada. We have also tried to use only those features of Maple, which are commonly used by a typical beginning level user of CAS.

In the following pages we'll present two examples to demonstrate our work. For a complete list of the projects, the reader is encouraged to contact the authors. For related background information on the use of Maple, we encourage you to consult the authors' earlier papers [1], [2] and [3].

1 An Elusive Limit

Following is a problem which appears in Stewart's multivariable calculus text [4] as a laboratory project. Students are asked to estimate or determine the limit of a function when x approaches zero numerically, graphically, using L'hospital's Rule and using Taylor Series expansion. We'll use Maple to perform all the tasks.

```
> fn:=x->sin(tan(x))-tan(sin(x)); fd:=x->arcsin(arctan(x))-
arctan(arcsin(x));
```

$$fn := x \rightarrow \sin(\tan(x)) - \tan(\sin(x))$$

$$fd := x \rightarrow \arcsin(\arctan(x)) - \arctan(\arcsin(x))$$

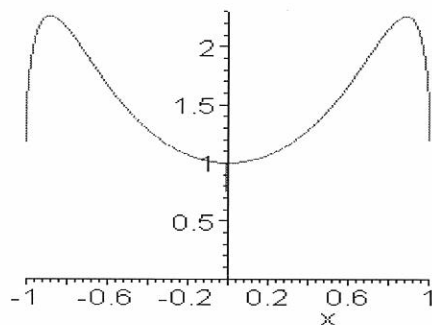
```
> # Estimate the limit as x approaches 0
```

```
> fn(0.01)/fd(0.01);fn(0.001)/fd(0.001);fn(0.0001)/fd(0.0001);
2.000000000 3.333333333 3.333333333
```

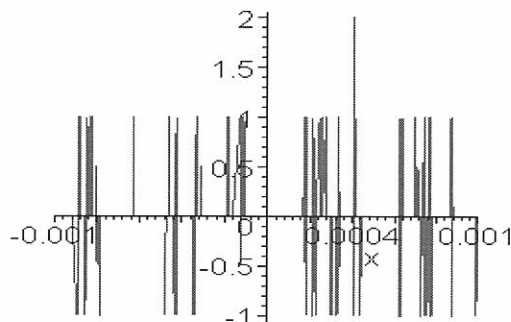
```
> fn(-0.01)/fd(-0.01);fn(-0.001)/fd(-0.001);fn(-0.0001)/fd(-0.0001);
2.000000000 3.333333333 3.333333333
```

One might suspect that the limit is $1/3$. However, the first graph implies that the limit is 1 and the second graph is inconclusive:

```
plot(fn(x)/fd(x),x=-1..1);
```



```
plot(fn(x)/fd(x),x=-0.001..0.001);
```



Let's use L'Hospital's Rule. We need to repeat this rule seven times. For the sake of brevity we'll only show the leading terms in derivatives.

We need to differentiate Numerator and Denominator seven times to get a result.

```
> fnprime7:=diff(fn(x),x$7);
```

$$\begin{aligned} \text{fnprime7} := & -1824 \tan(\sin(x))^4 (1 + \tan(\sin(x))^2)^2 \cos(x)^7 \\ & + 5712 (1 + \tan(\sin(x))^2)^3 \cos(x)^5 \tan(\sin(x)) \sin(x) + (1 + \tan(\sin(x))^2) \cos(x) \\ & - 182 (1 + \tan(\sin(x))^2)^2 \cos(x)^3 \\ & \dots\dots\dots \end{aligned}$$

```
> fdprime7:=diff(fd(x),x$7);
```

$$\begin{aligned} \text{fdprime7} := & \frac{60480 \arcsin(x)^3 x^3}{(1-x^2)^5 (1+\arcsin(x)^2)^4} - \frac{9450 x \arctan(x)}{(1+x^2)^7 (1-\arctan(x)^2)^{(7/2)}} \\ & + \frac{60480 \arcsin(x)^2 x^2}{(1-x^2)^{(9/2)} (1+\arcsin(x)^2)^4} + \frac{10080 \arcsin(x)^2}{(1-x^2)^{(7/2)} (1+\arcsin(x)^2)^4} \\ & - \frac{46080 \arcsin(x)^6}{(1-x^2)^{(7/2)} (1+\arcsin(x)^2)^7} - \frac{720}{(1+x^2)^4 \sqrt{1-\arctan(x)^2}} \\ & + \frac{17280 x^2}{(1+x^2)^5 \sqrt{1-\arctan(x)^2}} + \frac{784}{(1+x^2)^5 (1-\arctan(x)^2)^{(3/2)}} \dots\dots\dots \end{aligned}$$

```
> limit1at0:=evalf(subs(x=0,fnprime7))/evalf(subs(x=0,fdprime7));
limitat0 := 1.000000000
```

Let's expand numerator and denominator about x=0 using Taylor series

```
> ntaylor:=convert(taylor(fn(x),x=0,12),polynom);
```

$$\text{ntaylor} := -\frac{1}{30}x^7 - \frac{29}{756}x^9 - \frac{1913}{75600}x^{11}$$

```
> dtaylor:=convert(taylor(fd(x),x=0,12),polynom);
```

$$\text{dtaylor} := -\frac{1}{30}x^7 + \frac{13}{756}x^9 - \frac{2329}{75600}x^{11}$$

```
> limit2at0:=subs(x=0,simplify(ntaylor/dtaylor));
```

```
limit2at0 := 1
```

Finally let's use Maple's limit command:

```
> limit3at0:=limit(fn(x)/fd(x),x=0);
```

```
limit3at0 := 1
```

2 Limit of Fibonacci sequence

Maple can also be used with less advanced students. We have been successful in introducing some pre-service teachers to Maple and encouraging them to perform some mathematics that they otherwise would not attempt.

Many people who would assert that they are not interested in mathematics will show an interest in the Fibonacci sequence and the Golden Ratio. We believe that the popularity of these topics is due to the numerous simple connections that can be suggested between the Fibonacci sequence, the Golden Ratio, the way plants grow, generation of rabbits, and our innate sense of beauty¹.

The Fibonacci sequence is defined recursively by $F(1) = 1$, $F(2) = 1$, and $F(n) = F(n-2) + F(n-1)$. The Golden Ratio, ϕ , is defined as the positive root of $x^2 - x - 1 = 0$. It has been observed that in many natural products you can find successive terms of the Fibonacci Sequence, for example the central seed arrangement of a sunflower can be seen to form a clockwise and an anticlockwise set of spirals. Counting the spirals in an example from the garden we find 21 clockwise spirals and 34 anticlockwise spirals. And these are two successive terms of the Fibonacci Sequence. It is widely reported that the ratio of width of the Parthenon to its height is the Golden Ratio. For even more fun one can investigate the ratio of $F(n+1)$ to $F(n)$ which appears to get closer and closer to the Golden Ratio which is approximately 1.61803398875. Using the first few terms of the Fibonacci sequence we observe:

$1/1=1$, $2/1=2$, $3/2=1.5$, $5/3=1.67$, $8/5=1.6$, $13/8=1.62\dots$ and the claim begins to look plausible, but five ratios do not make a limit. This is a good time to introduce the student to limit concepts. How do you take a limit of a recursively defined function? Left overnight with this question one of our students returned with Binet's Formula

for the n^{th} term of the Fibonacci sequence. If $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$ then

$$F(n) = \frac{a^n - b^n}{a - b} = \frac{a^n - b^n}{\sqrt{5}}. \text{ Now Maple can help.}$$

```
> a:=(1+sqrt(5))/2;b:=(1-sqrt(5))/2;  
> F:=n->(a^n-b^n)/sqrt(5);  
> Q:=n->F(n+1)/F(n);  
> limit(Q(x),x=infinity);
```

For the sake of brevity the Maple response to the first three equations has been omitted, the response to the final Maple equation is:

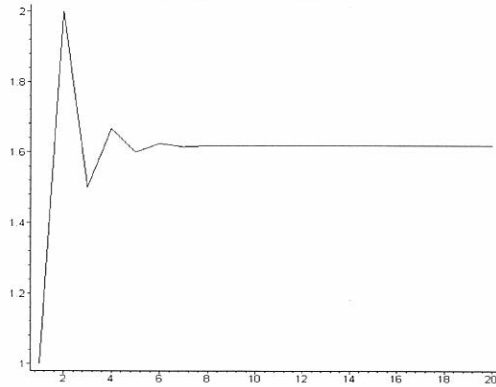
$$\frac{1}{2} + \frac{\sqrt{5}}{2} \quad \text{Which is the exact value of } \phi, \text{ the Golden Ratio.}$$

This simple exercise gave us a chance to get the students to think a bit about how Maple works. We asked them to graph Q so that they could “see” the convergence to ϕ .

¹ Keith Devlin (MAA Online, June 2004, “Devlin’s Angle”) expresses serious doubt about the frequent occurrence of the Golden Ratio in architecture and the human psyche.

> `plot(Q,1..10,1..3,thickness=1);` The resulting plot was empty. After the hint that Maple was looking at Q as a function of a continuous variable they evaluated Q for some non-integer values and the mystery was solved. Then, together, we found a way to visualize the convergence of Q .

> `data:=seq([n,Q(n)],n=1..20);`
 > `plot([data],style=line);`



We firmly believe that when used in conjunction with sound instruction, Maple is a valuable tool in the undergraduate classroom. It can be used to help solve some tough problems and to awaken interest in certain students by helping them with some objectively easy problems.

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