

DIFFERENTIAL EQUATIONS ON THE VOYAGE 200 AND TI-89

Al Stickney
Wittenberg University
Dept. of Math & Comp. Sci.
Springfield, OH 45501
astickney@wittenberg.edu

The purpose of this paper is to familiarize the reader with those features and capabilities of the Voyage 200 and TI-89 calculators that have a direct impact on the teaching of a standard first undergraduate course in ODE. Mathematically, both calculators have the same functionality. Any differences are due to differences in the keyboards and the screens. This paper is based on version 3.10 of the Voyage 200 and TI-89 operating systems.

When working with differential equations on a TI-89, one should consider using a custom menu containing the more common words, names, and commands in order to avoid excessive use of the "alpha" keyboard. A custom menu is usually defined using a program, and an example of such a program is available from the author.

SECTION ONE: THE DESOLVE COMMAND

The deSolve command ("C" in the Calculus menu) can be used to find symbolic solutions of differential equations. Its use is limited to a single first- or second-order equation. It can be used to find general solutions or to find particular solutions of initial-value and boundary-value problems. Of course, it cannot solve all such problems.

The "prime" operator ([2nd] [B] on a 200 and [2nd] [=] on an 89) is used to enter the differential equation. Use two "prime" symbols to indicate a second derivative. The format for finding a general solution is:

deSolve(D.E. , indep var, dep var).

An example of this format is deSolve($y'' + 2y' - 3y = \sin(x)$, x , y). Note that arbitrary constants are labeled using the @ symbol. The format for finding a particular solution is:

deSolve(first-order DE and condition , indep var, dep var), or

deSolve(second-order DE and condition , indep var, dep var), or

deSolve(second-order DE and condition1 and condition2 , indep var, dep var).

For a 2nd-order equation, if there are two conditions, they must be either the values of the function and its derivative at a common point (a "standard" initial value problem) or the value of the function at two different points. Valid examples would include:

deSolve($p' + 2p \cdot q = 0$ and $p(1) = 2$, q , p),

deSolve($y'' + 2y' + 5y = x$ and $y'(0) = 7$, x , y),

deSolve($3y'' - 2y' - y = e^x$ and $y'(0) = 1$ and $y(0) = 2$, x , y), and

deSolve(3 y'' - 2 y' - y = 0 and y(3) = 1 and y(0) = 2 , x , y).

Occasionally, deSolve will return an implicit solution when an explicit solution is possible. In this case, the explicit solution can usually be found using the "solve" command (from the algebra menu), although it may be necessary to make some assumptions as to the range of values for one or both of the variables (e.g. $x > 0$). More frequently, the solution returned by deSolve will be in a more complicated form than necessary and some algebraic manipulation (on the calculator) will be required to simplify the answer.

Basically, the deSolve command will solve almost all of the standard first- and second-order equations students are expected to solve analytically in a first course in ODE (second-order Cauchy-Euler equations seem to be an exception). The reader is encouraged choose a standard text and to use the deSolve command on randomly selected problems from the sections on separable equations, homogeneous equations, exact equations, first-order linear equations, second order linear equations with constant coefficients, the method of undetermined coefficients, and variation of parameters. Generally speaking, if the students are expected to solve an equation analytically (and it isn't higher than second order), deSolve will solve it.

It's important to understand that the arbitrary constants denoted with the @ symbol can be treated just like other variables in many ways. They can be used as the unknown in solve commands and they can be given temporary values using the "with" operator ([]). They cannot be assigned a permanent value using "define". The @ symbol itself is entered by pressing [2nd][R] on a 200 and [♦] [STO▶] on an 89.

As an example, suppose the general solution is found for $y'' + y = 0$ with the constants labeled @1 and @2. After defining $y(x)$ to be that solution the following sequence can occur:

entry: solve(y(0) = 5 and y($\pi/2$) = 7 , { @1, @2 })

answer: @1 = 5 and @2 = 7

entry: y(x) | @1 = 5 and @2 = 7

answer: 5 cos(x) + 7 sin(x)

This gives the particular solution to the corresponding boundary-value problem. Arbitrary constants are labeled with an increasing set of labels (@1, @2, @3, @4, etc.) every time one is generated by a new command until the counter is reset. Executing a NewProb command resets the counter.

SECTION TWO: THE ODE GRAPHER

The ODE grapher is designed to handle a system of up to 99 first-order D.E. The independent variable is always t and the dependent variables are called $y1'$, $y2'$, ..., $y99'$. To use the grapher with a higher-order D.E., it must be converted to an equivalent system of first order equations in the usual way (see example 2 below). Solutions can be found using either of two numerical methods: a modified Runge-Kutta method or Euler's method. In most circumstances, the Runge-Kutta method is the preferred method. Euler should only be used if accuracy is not as important, and speed is a primary concern.

A graph of the solution may be displayed in a variety of formats. The grapher can generate slopefields for 1-dimensional problems and direction fields for 2-dimensional problems. It can also plot multiple solutions based on interactively chosen initial conditions. Numerical solution data is available through TRACE and TABLE, and it can also be accessed on the home screen or stored in a data variable to generate lists and/or statistics plots.

An ODE Graphing Check-List

To put the calculator in **DIFF EQUATIONS** mode, press [MODE] and choose [6] in the graph options menu.

1. Press [Y=] and enter the differential equation(s) and any initial conditions. Initial conditions are not necessary if you want to graph a slopefield or direction field without a particular solution. On this screen, you may also select which equations will be graphed and what "style" will be used for each graph.
2. While still on the [Y=] screen, press [F1][9] or [♦][F] ([♦][1] on an 89) to access the "graph formats" dialog. Be sure that the "solution method" and "fields" options are set as desired.
3. Leave the "graph formats" dialog and return to the [Y=] screen. Unless the "SLPFLD" option is selected, press [F7] to access the "axes" dialog and choose which variable(s) will be represented by the horizontal and vertical axes in the graph.
4. Press [WINDOW] to select the "window" parameters. These parameters include the boundaries of the xy-screen and the t -interval to be used for the solution. Depending on other settings, there are 2 to 4 other parameters at the bottom of this screen. Note that normally, "ncurves" should be set to zero.
5. Select [GRAPH] to graph the solution.

Example 1

Graph the solution of the differential equation $y' = -xy$, with initial condition $y(0)=2$.

In **DIFF EQUATIONS** mode, and go to the [Y=] screen. Remember that the independent variable is t and we can make the dependent variable $y1$. Enter the equation as $y1' = -t*y1$. Enter the initial condition by setting $t0 = 0$ and $y11 = 2$. ($t0$ can also be set on the WINDOW screen.) Make sure the statistics plots (Plot1, etc., located above $t0$) are off, and select the "Line" style for $y1'$ in the Style menu ([F6]). Note the "check-mark" next to $y1'$ means it is "selected". (See Figure 1 below.)



Figure 1



Figure 2

Next press [F1][9] or [◆][F] ([◆][1] on an 89) to access the "graph formats" dialog, and set the solution method to "RK" and the fields option to "FLDOFF" (Figure 2). Return to the [Y=] screen and press [F7] to go to the "axes" dialog and set the axes option to "TIME" (Figure 3). Lastly, set the values on the WINDOW screen to match those shown in figure 4.

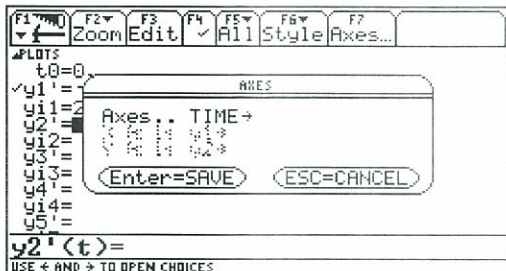


Figure 3

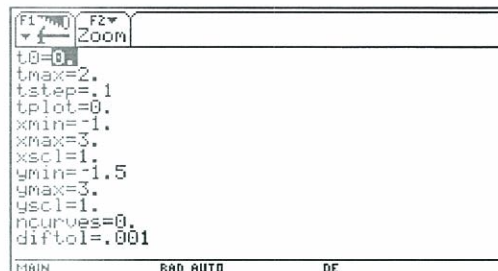


Figure 4

Now press GRAPH to obtain the graph shown in figure 5.

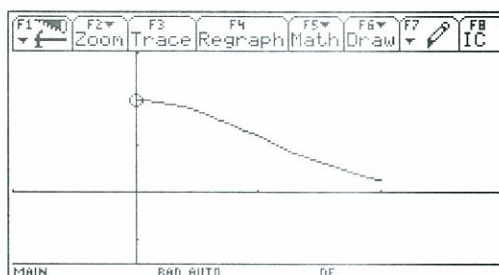


Figure 5

Try the following activities with this differential equation:

- Use TRACE to see numerical values corresponding to the solution. You can move to new t -values using the left/right arrow keys, or by typing in the desired t -value and pressing ENTER. (The value of t must be in the interval specified by t_0 and t_{max} .)
- Go to the home screen, and type $y1'(1.75)$ to see the value of the solution at $t = 1.75$. Note that this is very unfortunate notation, since this is really $y1(1.75)$, not $y1'(1.75)$.
- Go to the TblSet dialog and set $tblStart = -0.3$ and $\hat{t}tbl = 0.15$. Press TABLE to see a table of values for the solution. Note that there is no limit on the t -values in the table.
- Go back to the graph and superimpose the analytic solution $y = 2e^{(-x^2/2)}$ using DrawFunc in the DRAW menu. Following the command, just type the function (without the "y=") following the command and press ENTER.
- Clear the function drawn in (d) using ClrDraw (in the Draw menu), then go to the "graph formats" dialog, and select SLPFLD. Notice that the solution is now graphed across the entire slopefield and the values of t_0 and t_{max} are ignored..
- Try deleting the initial condition on the [Y=] screen and regraphing. The grapher will graph just the slopefield without any solution. (Note: When FLDOFF is selected, initial conditions must be specified unless the value of "ncurves" is at least 1.)

- (g) With the slopefield on the screen, choose IC ([F8]). Move the cursor to any point on the screen, and press ENTER to graph the solution which has that point as its initial condition. Now choose IC again and this time, enter the initial conditions by typing the values for t and y1. This process can be repeated indefinitely to add additional solutions to the screen. Curves drawn using IC cannot be traced. The style of these curves can be changed by setting the style for y1' on the [Y=] screen. Try using "THICK".

Example 2

Investigate the D.E. $y'' + xy' + 2y = \sin x$, with $y(0)=1$ and $y'(0)=-1$.

First, the equation must be converted to an equivalent system of two first-order equations. This is accomplished by introducing a new variable $w = y'$. We then have:

$$y'' = -xy' - 2y + \sin x \quad \text{or,} \quad w' = -xw - 2y + \sin x, \quad \text{with } y(0)=1 \text{ and } w(0)=-1.$$

Translating (x becomes t, y becomes y1, and w becomes y2), the equations to enter are:

$$\begin{aligned} y1' &= y2 & (y' &= w) \\ y2' &= -t \cdot y2 - 2y1 + \sin t & (w' &= -xw - 2y + \sin x) \end{aligned}$$

where the initial values are $t_0 = 0$, $y_{i1} = 1$, and $y_{i2} = -1$. Enter these equations and initial conditions on the [Y=] screen as shown in figure 6. Use [F4] to deselect y2 so it won't graph. Set the style for y1 equal to "Line". Go to the "graph formats" screen and select FLDOFF. Check that the TIME option is selected in the Axes menu.



Figure 6

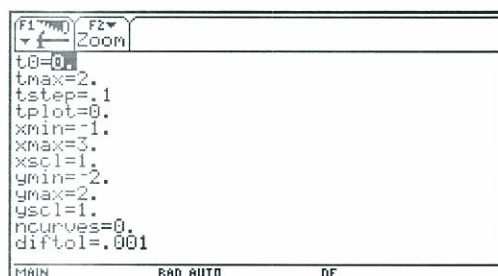


Figure 7

On the WINDOW screen, enter the values shown in figure 7. Now press GRAPH. The resulting graph (figure 8 below) is the graph of the solution as a function of the independent variable. In the original notation of the problem, it is a graph of the solution $y(x)$ versus x .

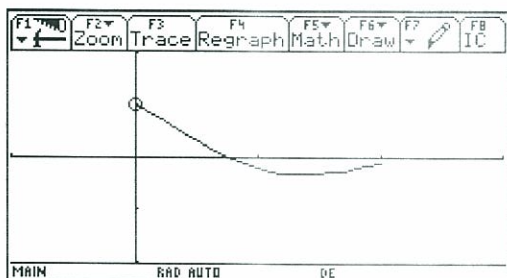


Figure 8

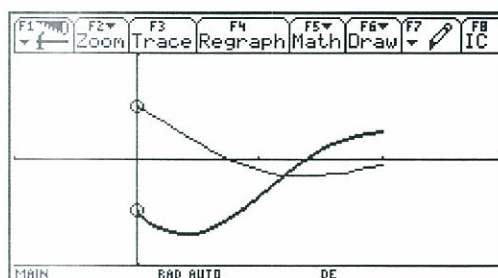


Figure 9

Only y_1 is graphed because only y_1' was selected on the [Y=] screen. Even if both y_1' and y_2' were selected, you can get the same graph by choosing the CUSTOM option in the Axes menu and choosing "t" and "y1" for the two axes. To investigate further, try the following:

- With the TIME option still selected in the Axes menu, return to the [Y=] screen and select y_2' in addition to y_1' . Check that the style for y_2' is set for "Thick" (the default) and regraph. Both y_1 and y_2 will be graphed as functions of t . In other words, this is a graph of the solution and its derivative. (See figure 9.) Both graphs can be traced.
- Note that if you use IC ([F8]) in this setting, you are asked what the axes represent. Normally, one would leave the x-axis set to "t" to avoid changing the meaning of the axes. For the y-axis you can choose either y_1 or y_2 . If the y-axis is set to y_1 , you may then choose new initial conditions for t and for y_1 , and the resulting solution y_1 is plotted. A corresponding graph for y_2 is not plotted. If the y-axis is set to y_2 , then new initial conditions may be set for t and y_2 and the resulting y_2 will be plotted, without a corresponding graph of y_1 .
- Go to the HOME screen and type the command "BldData RKData". This will compute the values for the original problem (using the initial values on the [Y=] screen) and place them in a data variable called RKData. (Any variable name may be used in place of RKData.) You can view these values using the Data/Matrix Editor (figure 10). The values may also be used to generate lists and statistics plots. Values for "t" and for all of the functions selected on the [Y=] screen are stored in the data variable.

	F1 Plot	F2 Setup	F3 Cell	F4 Header	F5 Calc	F6 Util	F7 Stat
DATA	t	y1	y2				
	c1	c2	c3	c4	c5		
1	0.	1.	-1.				
2	.1	.89069	-1.179				
3	.2	.76578	-1.311				
4	.3	.6382	-1.392				
5	.4	.48921	-1.42				
6	.5	.34801	-1.396				
7	.6	.21167	-1.325				

r1c1=0.

Figure 10

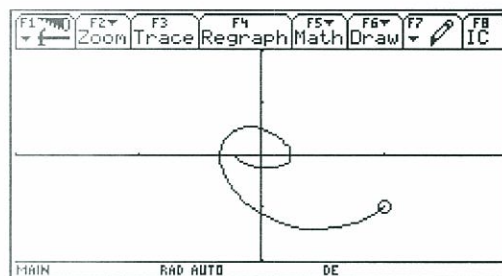


Figure 11

- Change the style for y_2' to "Line" and change the AXES settings to "y1" and "y2" for the x-axis and y-axis respectively. On the WINDOW screen, change the t -interval to [0,6] and the x-interval to [-2,2]. Press GRAPH to plot a trajectory in the phase plane. In the notation of the original equation, the resulting graph (figure 11) plots y' versus y .
- Use TRACE to find numerical values for this phase-plane graph.
- With the phase plane graph still on the screen, use IC to choose other initial conditions and generate trajectories for other solutions. Note that in this context, it's best to accept the settings offered for the axes choices (y_1 and y_2) so that you interactively select the initial values of y_1 and y_2 , but t_0 remains unchanged.
- If DIRFLD is selected in the graph formats menu, a direction field for y_1 versus y_2 is drawn. Since these equations depend explicitly on t , the shape of the direction field changes as t does. The value of **dtime** on the WINDOW screen determines which value of t is used to compute the direction field.

Example 3

Explore the rabbit-fox predator-prey model given by $F' = -F + 0.1F \cdot R$ and $R' = 3R - F \cdot R$.

Select FLDOFF in the graph formats menu, TIME for the axes option, and enter the values shown in figures 12-13 on the [Y=] and WINDOW screens. Be sure to "select" both y1 and y2 for graphing, and choose different "styles" for the two graphs.

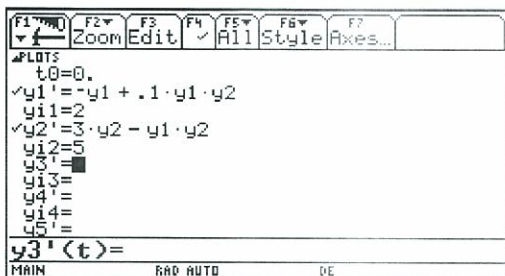


Figure 12



Figure 13

Now press GRAPH to graph the two populations over time. This graph is shown in figure 14 below. Next delete the initial values y1 and y2 on the [Y=] screen, select DIRFLD in the graph formats menu, and press GRAPH to display the direction field in the fox-rabbit phase plane (figure 15). Note that selecting DIRFLD automatically sets the x-axis and y-axis to the defaults y1 and y2.

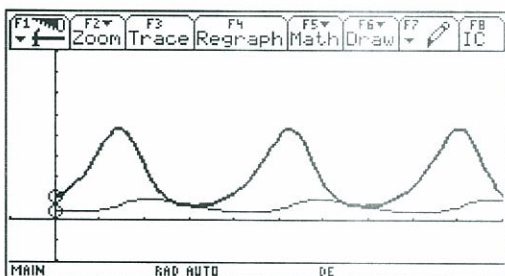


Figure 14

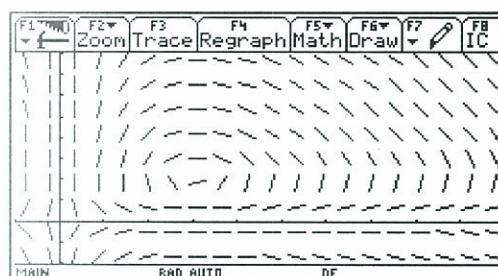


Figure 15

To graph a family of phase plane solutions in the direction field, go to the [Y=] screen and enter y1={4,5,6} and y2={8,16,24}. Also, change tmax to 5 on the WINDOW screen. Figure 16 shows the graph. TRACE can be used to find numerical values, and the free cursor can be used to estimate the coordinates of the equilibrium point (3 foxes and 10 rabbits).



Figure 16

SECTION THREE: USING ALGEBRA/CALCULUS CAPABILITIES

The symbolic capabilities of the calculator can be used to simplify some of the topics in a first ODE course where students tend to be overwhelmed by manipulative detail. The next two examples consider Cauchy-Euler equations that cannot be solved by the deSolve command.

Example 4

Consider $x^2 y'' + x y' - y = 0$. First, define $y(x) = x^m$, and then enter the D.E. into the calculator. This yields an equation which can be solved for "m". The sequence is as follows:

entry: Define $y(x) = x^m$

entry: $x^2 * d(y(x),x,2) + x * d(y(x),x) - y(x) = 0$

answer: $(m^2 - 1) \cdot x^m = 0$

entry: solve(ans(1), m)

answer: $m > 0$ and $x = 0$ or $m = -1$ or $m = 1$

Of course, one could simplify the last answer by dividing the equation by x^m before solving it or by adding the condition that x is non-zero to the solve command. The solution can also be found in a single step by entering the following command:

solve($x^2 * d(x^m,x,2) + x * d(x^m,x) - x^m = 0, m$) | $x \neq 0$.

Obviously, the first approach is much more instructive than the second, but the second is more efficient once the technique is understood.

Example 5

Consider $x^2 y'' - x y' + y = 0$. The technique used in the previous example finds that $y = x$ is a solution of this equation, and that $m = 1$ is a repeated root of the auxiliary equation. Now use reduction of order to find the second independent solution.

entry: Define $y(x) = x * u(x)$

entry: $x^2 * d(y(x),x,2) - x * d(y(x),x) + y(x) = 0$

answer: $(d(u(x),x,2) \cdot x + d(u(x),x)) \cdot x^2 = 0$

This leads to the equation $x \cdot u'' + u' = 0$ which can be solved on the calculator in any number of ways (including simply using the deSolve command). Of course, one could introduce a new variable, $w = u'$ to emphasize that the problem has been reduced to solving a first-order equation.

Example 6

Find a particular solution of $y'' - y' + y = 2 \sin(3x)$ using the method of undetermined coefficients.

entry: Define $y(x) = a * \cos(3x) + b * \sin(3x)$

entry: $d(y(x),x,2) - d(y(x),x) + y(x) = 2 \sin(3x)$

answer: $(-8a - 3b) \cos(3x) + (3a - 8b) \sin(3x) = 2 \sin(3x)$

entry: $\text{solve}(-8a - 3b = 0 \text{ and } 3a - 8b = 2, \{a,b\})$

answer: $a = 6 / 73 \text{ and } b = -16 / 73$

Of course, the deSolve command can find the general solution for this equation (although the solution is not expressed in a very simple form). Note however that the above technique can be used with higher order equations as well.

Example 7

Find a series solution of $y'' + y \cos(x) = 0$. Find the terms up to degree 5.

entry: Define $y(x) = a + b x + c x^2 + d x^3 + f x^4 + g x^5$

entry: $d(y(x),x,2) + y(x) * \text{taylor}(\cos(x),x,5) = 0$

answer: $\dots + (-a/2 + c + 12f) x^2 + (b+6d) x + a + 2c = 0$

Recalling that we expect two arbitrary constants, we begin with the constant term and attempt to write c, d, f, and g in terms of a and b.

entry: Define $c = -a / 2$

entry: (pull down the long answer above)

answer: $\dots + (b + 6d) x = 0$

entry: Define $d = -b/6$

entry: (pull it down again)

answer: $\dots\dots + (12f - a)x^2 = 0$

entry: Define $f = a/12$

entry: (pull it down again)

answer: $\dots\dots + (20g - 2b/3)x^3 = 0$

entry: Define $g = b/30$

entry: (pull it down again)

answer: $\dots\dots + 3ax^4/8 = 0$

We have written c, d, f, g in terms of a and b , so we are done with the process. To find the first few terms of two independent solutions, do the following:

entry: $y(x) \mid a = 1 \text{ and } b = 0$

answer: $x^4/12 - x^2/2 + 1$

entry: $y(x) \mid a = 0 \text{ and } b = 1$

answer: $x^5/30 - x^3/6 + x$

One can now choose initial conditions, use the ODE grapher to plot a numerical solution, and then use DrawFunc to superimpose the graph of the corresponding series solution.

In addition to the examples above, the CAS capabilities of the Voyage 200 and the TI-89 can help with much of the manipulation involved with topics such as the Laplace transform and Fourier series. The calculator can also manipulate symbolic matrices and determinants, and find eigenvalues and eigenvectors. These features are useful in the study of systems of equations.