

## DUELING (DUALING) SOLIDS: ENHANCING STUDENT AND TEACHER GEOMETRICAL UNDERSTANDING WITH CABRI 3D

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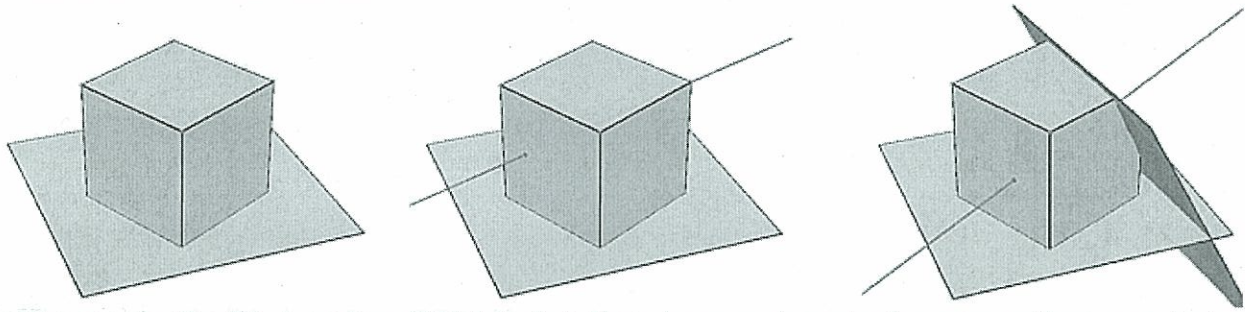
The introduction of two-dimensional dynamic geometry software (DGS) in school classrooms has fundamentally transformed the way that many teachers and students approach the teaching and learning of *plane* geometry. In particular, tools such as *Geometer's Sketchpad* and *CABRI Geometry II* have enabled users to reconsider geometry as an inquiry-oriented discipline, one in which students become active participants and creators - rather than passive consumers - of mathematics.

The recent advent of powerful *three-dimensional* DGS (DGS-3D) promises to transform the study of solid geometry in much the same way. With the tools, many of the benefits previously realized with DGS are now accessible to students studying geometric objects in three-dimensions. Three-dimensional visualization software, such as *CABRI 3D Version 2* (Cabrilog, 2006), provides at least three instructional benefits to students and teachers. First, the software provides users with opportunities to extend their understanding of familiar three-dimensional figures by allowing them to uncover relationships among shapes that would otherwise be difficult to explore. Secondly, the measurement capabilities of the software provide opportunities to connect solid geometry to algebra and proof-writing. Lastly, three-dimensional visualization software affords teachers and students with enhanced opportunities to extend generalizations from two to three dimensions. We explore each of these instructional benefits in the remainder of this document.

### Discovery of Relationships Among Familiar Figures

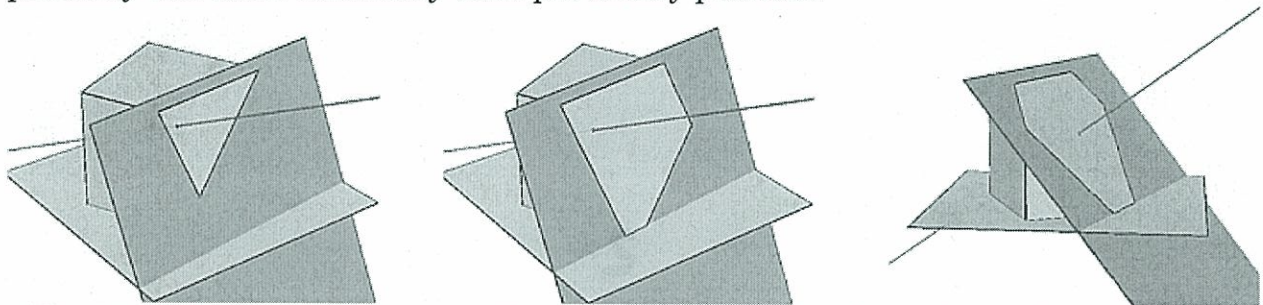
Cube Slices. Exploring cross sections of three-dimensional shapes is a familiar activity for many students. Traditionally, visualizing cross sections in three dimensions is a difficult task for students. The *Principles and Standards of School Mathematics* (NCTM, 2000) notes that students who have little experience with spatial-visualization activities typically experience difficulty visualizing cross sec-

tions without a model (p. 237). While physical materials (e.g. cubes of cheese, hollow plastic models) prove helpful as visualization tools, the expense and crudeness of these materials limit their utility. They lack the precision required to make clear-cut conjectures. On the other hand, DGS-3D allows students to generate thousands (if not millions) of precise cross-sections in a fraction of the time required with physical models. Figure 1 illustrates steps required to construct dynamic cube cross sections with CABRI 3D.



**Figure 1:** (Left) A cube; (Middle) A line is constructed connecting a point on a cube face with an opposing vertex; (Right) Students construct a perpendicular cutting plane.

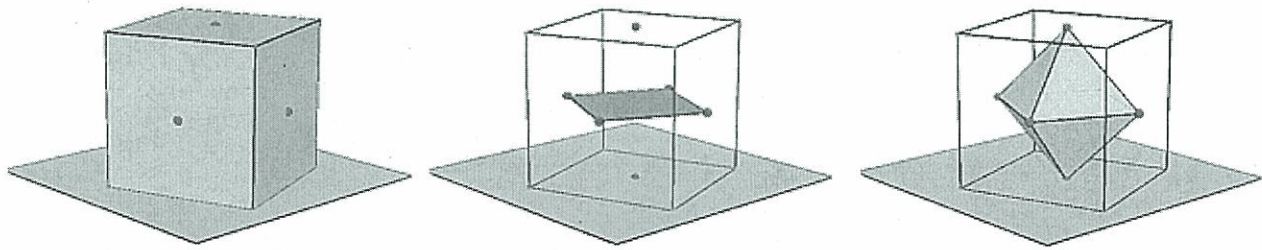
Capitalizing on the dynamic nature of CABRI 3D, students drag points in their sketches to generate numerous examples of cross sections - such as those depicted in Figure 2. A previously problematic activity, cutting cubes, is now explored more precisely and more efficiently than previously possible.



**Figure 2:** Dynamic cross-sections constructed by dragging objects within CABRI 3D.

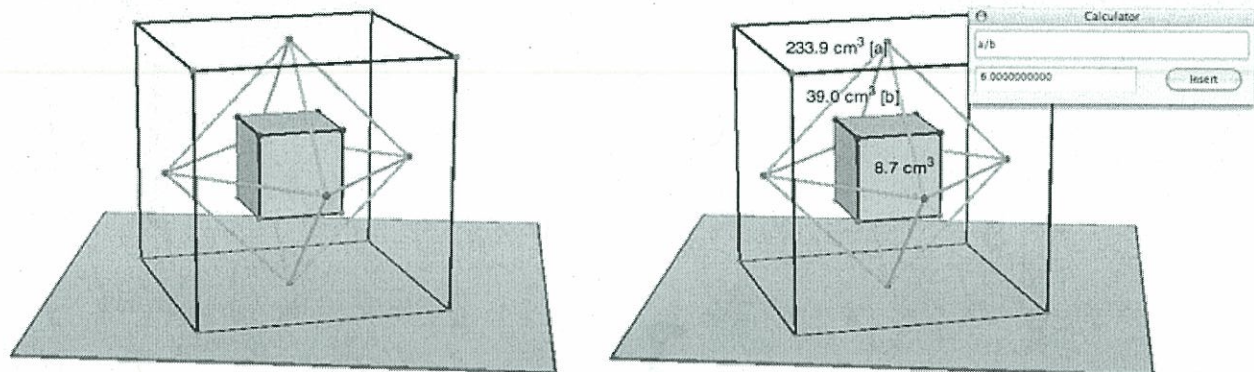
Alternating Dual Pairs. Many students first encounter geometric duals as they study Platonic solids. CABRI 3D enables students to quickly construct duals of various polyhedra. Figure 3 illustrates the construction of a cube dual. Centers of each face of the cube are connected with line segments to create the dual.





**Figure 3:** (Left) Center of each cube face is constructed; (Middle) Centers of lateral faces connected to form polygon (square); (Right) The dual of the cube (octahedron) is constructed.

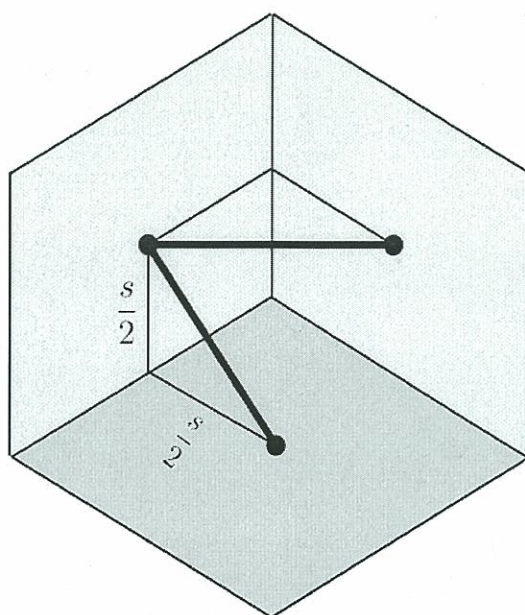
While students can create duals of various shapes with physical materials (e.g. Polydrons; Zome tools), the creation of “alternating dual pairs” (e.g. duals of duals) is problematic with such materials. With CABRI 3D, however, the task is far more accessible to students. Furthermore, the measurement capabilities of CABRI 3D enable students to explore heretofore unfamiliar numerical relationships that exist within sequences of duals. For instance, the sequence of volumes created by cube-octahedron duals is at a ratio of 6:1. This result is suggested in Figure 4.



**Figure 4:** (Left) Alternating cube-octahedron duals constructed; (Right) Volume of each dual is calculated along with ratios of these volumes.

### Connecting Three-Dimensional Geometry to Algebra and Proof

Dual Proofs. DGS provides students and their teachers with entirely new approaches for exploring proof (DeVillers, 1999; Hanna, 2000). The measurement capabilities of DGS-3D provide opportunities to form conjectures about numerical relationships that can be proved algebraically. For instance, students may use algebra to prove that the ratio of the volume of a cube to its dual is always 6:1.



Edge of octahedron =

$$\sqrt{\left(\frac{s}{2}\right)^2 + \left(\frac{s}{2}\right)^2} = \frac{s}{\sqrt{2}}$$

Area of square connecting consecutive centers of lateral faces:

$$\frac{s}{\sqrt{2}} \cdot \frac{s}{\sqrt{2}} = \frac{s^2}{2}$$

Volume of square pyramid (half of octahedron):

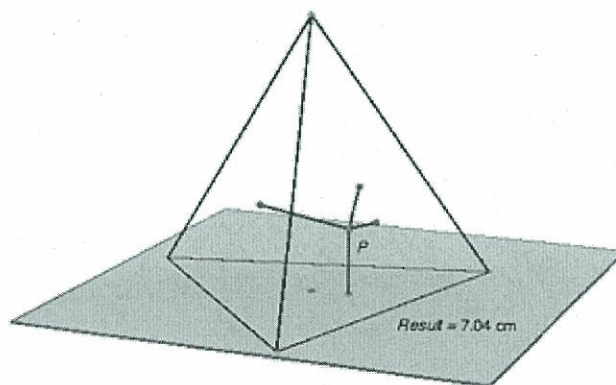
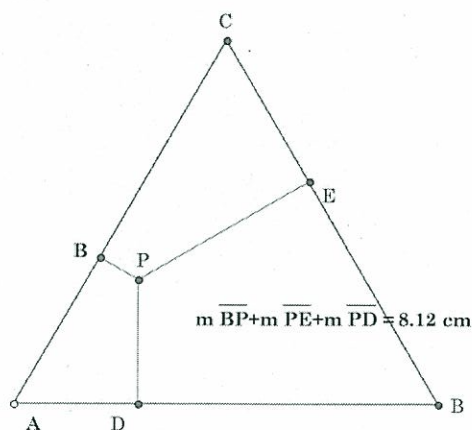
$$\frac{1}{3}Bh = \frac{1}{3} \left(\frac{s^2}{2}\right) \left(\frac{s}{2}\right) = \frac{s^3}{12}$$

So volume of octahedron is  $\frac{1}{6} \cdot (\text{Volume of Cube})$

**Figure 5: Student proof of cube-octahedron dual conjecture.**

### Extending Generalizations to Three Dimensions

Perhaps most intriguing use of DGS-3D is its capability for exploring generalizations from two to three dimensions. For instance, students can explore which triangle centers (e.g. orthocenter, incenter) generalize to three dimensions as centers of triangular pyramids (Oldknow, 2006). Figure 6 (left) illustrates Viviani's Theorem as depicted in *Geometer's Sketchpad*. The theorem states that for a point P inside an equilateral triangle ABC, the sum of the perpendiculars from P to the sides of the triangle is constant, equal to the altitude  $h$ . Figure 6 (right) suggests a three-dimensional analog to the theorem with arbitrary point inside a tetrahedron.



**Figure 6: (Left) Viviani's Theorem as modeled with Geometer's Sketchpad; (Right) A 3D analog to the theorem modeled with CABRI 3D.**



## References

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