EXPLORING CALCULUS USING INNOVATIVE TECHNOLOGY

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For the past decade, our Mac-only software package, TEMATH, has been available as a free download from our web site

www2.umassd.edu/TEMATH

Over the past few years, we have been converting this software package to one that runs on computers with Macintosh OS X, Linux, or current versions of Microsoft Windows/NT/XP operating systems that have a Java JVM 1.4.2 (or later installed). We have integrated into the software a robust set of visualization, modeling, demonstration, numerical, and graphical tools. These tools have been class tested in both our single-variable and multivariable calculus classes.

The goals of the software are to:

- Enhance the teaching and learning of calculus,
- Make it easier for teachers to do real applied mathematics in the classroom,
- Make mathematics enjoyable for all students,
- Increase the active participation of students in the class,
- Provide a visual environment for the presentation of mathematical concepts,
- Give teachers an easy-to-use set of tools for demonstrating the theorems of calculus,
- Include a friendly interface for importing digital images and movies that can be used for generating data which will be modeled by students, and
- Allow students to easily model everyday physical phenomenon.

Our software gives the instructor another set of technological tools that can be used in the classroom to improve the presentation of calculus. The software is free to all and the new version will be available for download from our web site this summer.

We teach calculus as part of the IMPULSE first-year engineering program at UMass Dartmouth. In this program, all classes (Engineering, Physics, and Calculus) are taught in an active learning environment where students work together in teams. A typical class consists of about 40-48 engineering students working in 10-12 teams of four students each. Each team sits at a large table with two computers and each class period is two hours long. Since we teach in the same room as physics and engineering, we have a variety of equipment available to us for performing experiments. Below, we present a set of calculus examples that demonstrate some of the features of TEMATH.

Hooke's Law

An applied problem that appears in all calculus books is Hooke's law for spring-mass systems. Hooke's Law states that the spring force is given by $\boldsymbol{F} = -\boldsymbol{kx}$, where x measures the displacement of the end of the spring from its relaxed position, and the spring constant k is a measure of the stiffness of the spring. To help convince our students that this law really works, we took a sequence of ten digital pictures of a mass hanging from a soft spring where the mass varied from 10 grams up to 100 grams in steps of 10 grams. Importing these images into TEMATH, we were able to play them as a movie. By simply watching this movie, it was quite easy to observe a constant downward motion of the spring indicating a linear relationship between the amount of mass and the stretch of the spring. However, to quantify this relationship, we were able to use TEMATH's Point tool to measure the displacement in the spring for each added increment of mass. This data along with its excellent linear fit is shown in Figure 1. We repeated the process described above for a stiff spring and that data and fit are also shown in Figure 1.

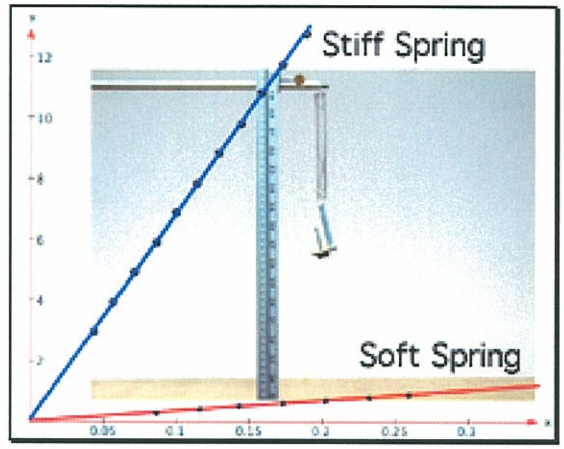

Figure 1 Hooke's Law

The Hyperbolic Egg

One of our goals in calculus is to use real world examples (data) to motivate a particular topic of study. A hanging cable (catenary) is the typical applied problem when introducing hyperbolic functions. Since the catenary evenly distributes the weight of the cable, Smith and Minton[2] suggest in a problem in their calculus book that the shell of an egg can be modeled with a hyperbolic cosine function. To test this hypothesis and have another motivational example for our students, we took a digital picture of an egg, imported the image into our software, and fit the shape of the egg with $f(x) = -1.25\cosh(0.8x) + 1.25$. Figure 2 shows the excellent fit.

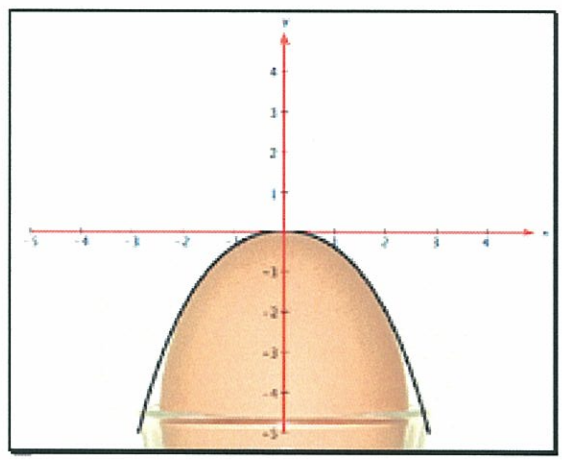

Figure 2 The Hyperbolic Egg

Snell's Law

TEMATH can be used to experimentally verify theoretical laws presented in calculus textbooks. For example, just about every first-year calculus book has some version of Snell's Law of Refraction in the section covering optimization. Students are usually asked to derive Snell's Law by showing the path of light waves from a point in one medium to a point in a second medium follows the path of minimum travel time. Using a standard presenter's laser pointer, we took a digital picture of the path of the laser light from air into water (See Figure 3) and imported it into TEMATH. For this experiment (see Figure 4), Snell's law states that

$$\text{Index of Refraction} = \boldsymbol{n} = \frac{\sin(\alpha)}{\sin(\beta)}$$

Using TEMATH's Vector Interface, we were able to measure the angles α and β. Substituting these values into Snell's law, we obtained the following estimate for the Index of Refraction for Water

$$\boldsymbol{n} = \frac{\sin(\alpha)}{\sin(\beta)} = \frac{\sin(0.8584)}{\sin(0.6021)} = 1.336$$

The known Index of Refraction for Water is $\boldsymbol{n} = \frac{4}{3} = 1.33333$.

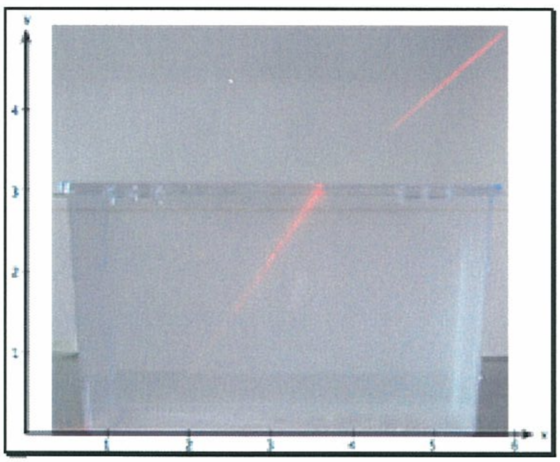

Figure 3 Snell's Law Experiment

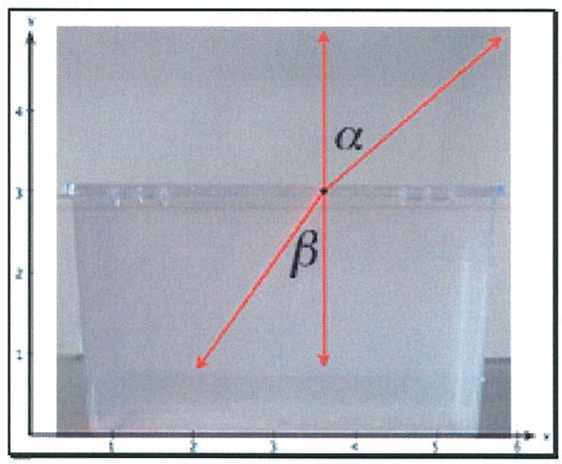

Figure 4 Snell's Law

Related Rates

Many calculus students have a difficult time solving related rates problems. Part of the difficulty is the visualization of the relationship between the rates of change of two distinct quantities. A typical related rate problem in a first-year calculus course is to find the rate of change of the angle of a camera tracking a rising rocket (or balloon) or a falling object. To help our students visualize this type of problem and to help them develop an intuition about related rates, we took a video of a dropped basketball. In the frames of the video, we superimposed the picture of a camera tracking the path of the ball (this was done for effect). We then imported the frames of this video into TEMATH. Stepping through the frames, one can observe the simultaneous motion of the ball and the camera. To empirically determine the relationship between these two rates of change, we used TEMATH's Line tool to measure the distance the ball fell between two successive frames of the video, and we used TEMATH's Vector tool to measure the change in the angle of the camera between the same two frames of the video. The theoretical relationship between the two rates of change is given by $\frac{d\theta}{dt} = \frac{\cos^2(\theta)}{k}\frac{dy}{dt}$, where y is the distance the ball has fallen in t seconds, k is the horizontal distance from the camera to the ball at time $t = 0$, and θ is the angle of the camera measured from the horizontal (see Figure 5). Since our camera takes a video at a rate of 30 frames per second, the time lapse between successive frames of the video is 1/30 of a second. For this small change in time, the theoretical relationship can be approximated by $\frac{\Delta\theta}{\Delta t} \approx \frac{\cos^2(\theta)}{k}\frac{\Delta y}{\Delta t}$.

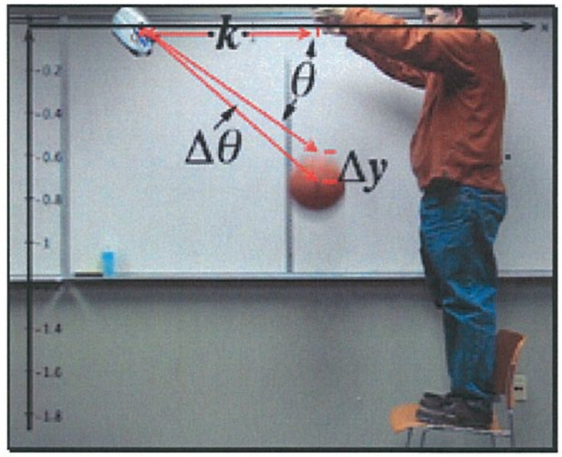

Figure 5 Related Rates Visualization

To check the accuracy of this model, we used TEMATH's tools to measure the following quantities:

k = 0.8409 meters, $\theta = 0.6826$ radians, $\Delta y = 0.1297$ meters, and $\Delta\theta = 0.0929$ radians

Substituting these values into the model, we get

$$\frac{d\theta}{dt} \approx \frac{\cos^2(\theta)}{k}\frac{\Delta y}{\Delta t} = \frac{\cos^2(0.6826)}{0.8409}\frac{(0.1297)}{1/30} = 2.786 \text{ radians/second}$$

This predicted value compares very well to the measured value $\frac{\Delta\theta}{\Delta t} = \frac{0.0929}{1/30} = 2.787$ radians/second. Experiments like these can be used to build students confidence in the mathematical models and algorithms that we present in a calculus class.

Vector Fields, Flow Lines, Contour Lines, Parametric Paths, and Line Integrals

We have added a flow line and contour line interface to vector fields, and some predator-prey models. Using the predator-prey model $< 2x - 1.2xy, -y + 0.9xy >$ as an example, students can study the behavior of solution curves by simply clicking a family of flow lines showing solutions for various initial conditions (see Figure 6). In Figure 7, we have drawn a family of flow lines and contour lines for the vector field $< y, x >$. Our vector field interface also includes a variety of parametric paths along which you can calculate a line integral. An interesting application that will surely motivate students is to use Green's Theorem to approximate the area of some well-know landmass. For example, if you want to approximate the land area of the oldest city park in the U.S., the Boston Common, obtain a satellite picture of the Boston Common from the web, and import this image into TEMATH. Then, using TEMATH's fitting tools to fit a piecewise smooth curve C_P to its boundary ∂P, the area of the Boston Common can be approximated by evaluating the line integral over this curve through the vector field $< -y, x > /2$ and

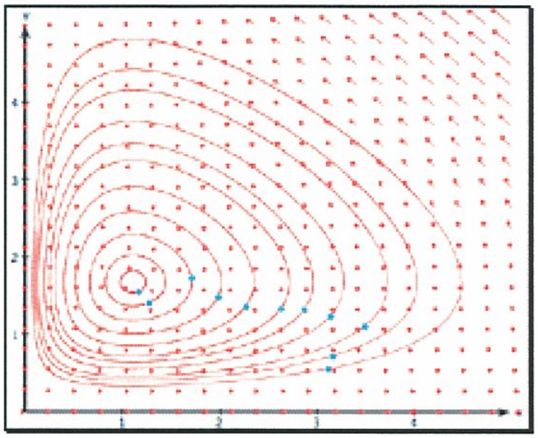

Figure 6 Predator-Prey Model

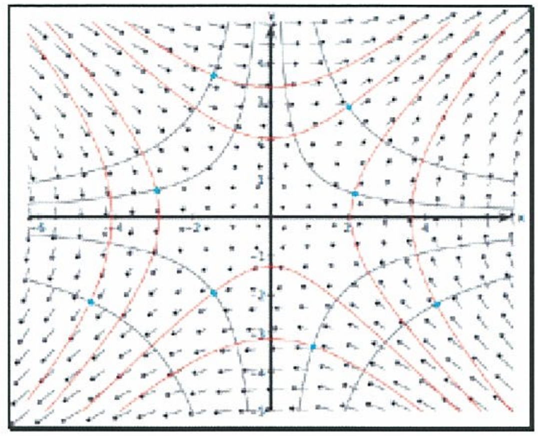

Figure 7 Flow Lines and Contour Lines

applying Green's Theorem:

$$\frac{1}{2}\int_{C_P}(xdy - ydx) = \frac{1}{2}\int_{\partial P}(xdy - ydx) \overset{Green's\,Theorem}{=} \int_P 1dA = \text{Area of Boston Common.}$$

Figure 8 shows the use of Green's theorem to calculate the land area of the Boston Common from a satellite map obtained from Google. Using this map and Green's theorem, we calculated the area to be 46.14 acres. The Boston Parks Dept. lists the area of the Boston Common as 46.45 acres! In this example, care needs to be taken to set the axes lengths to match the scale of the map.

Figure 8 Approximating the Land Area of the Boston Common

Bibliography

[1] Robert Kowalczyk and Adam Hausknecht, *TEMATH - Tools for Exploring Mathematics Version 2.1*, 2003.

[2] R. Smith and R. Minton, Calculus, 3rd Edition, McGraw Hill Higher Education, 2008.

[3] R. A. Serway, Principles of Physics, Harcourt Brace College Publishing, 1994.