SERIES SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH THE TI-89

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Power Series Solution about an Ordinary Point.

Suppose we are given a second order differential equation: $y'' + p(x)y' + q(x)y = 0$. We say that x_0 is an ordinary point for this equation if and only if both $p(x)$ and $q(x)$ are analytic x_0. That is, $p(x)$ and $q(x)$ can be represented by power series centered at x_0: $p(x) = \sum_{k=0}^{\infty} p_k (x - x_0)^k$ and $q(x) = \sum_{k=0}^{\infty} q_k (x - x_0)^k$, where both series have some positive radius of convergence. It is well-known that in this case the equation has two linearly independent power series solutions.

If $y = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ is a solution the equation, then this power series satisfies the initial conditions $y(x_0) = a_0$ and $y'(x_0) = a_1$. The coefficient of the second degree term can be found by solving the equation for y'' and substituting x_0:

$$y''(x_0) = -p(x_0)y'(x_0) - q(x_0)y(x_0) = -p(x_0)a_1 - q(x_0)a_0.$$

Therefore, $a_2 = \frac{y''(x_0)}{2!} = \frac{-p(x_0)a_1 - q(x_0)a_0}{2!}$. Repeated differentiation of the equation, substitution of x_0 and dividing both sides by $k!$ shows us:

$$a_k = \frac{y^k(x_0)}{k!} = -\sum_{j=0}^{k-2} \frac{k-1-j}{k \cdot (k-1)j!} p^{(j)}(x_0) a_{k-1-j} - \sum_{j=0}^{k-2} \frac{1}{k \cdot (k-1)j!} q^{(j)}(x_0) a_{k-2-j}$$

for $k = 2, 3, 4, \cdots$.

The program "ordinpt" performs the calculations above. The commands of this program are as follows:

```
:ordinpt(xo,yo,dyo,n)
:Prgm
:Local i,k
:0→i:0→k
:newList(n-2)→plist
```

```
:For i,1,n-2
:d(p(x),x,i)|x=xo→plist[i]
:EndFor
:newList(n-2)→qlist[i]
:For i,1,n-2
:d(q(x),x,i)|x=xo→qlist[i]
:EndFor
:newList(n)→a
:dyo→a[1]
:-(p(xo)*dyo+q(xo)*yo)/2→a[2]
:-p(xo)*a[2]/3-(plist[1]+q(xo))*a[1]/6-qlist[1]*yo/6→a[3]
:Local b,c
:0→b:0→c
:For k,4,n
:-p(xo)*a[k-1]/k-q(xo)*a[k-2]/(k*(k-1))→b
:-(plist[k-2]*a[1]+qlist[k-2]*yo)/(k!)→c
:b+c-1/(k*(k-1))*∑ (((k-1-j)*plist[j]*a[k-1-j]+qlist[j]*a[k-2-j])/(j!),j,1,k-3)→a[k]
:EndFor
:yo+∑ a[k]*(x-xo)^k,k,1,n)→ss
:EndPrgm
```

Before you run this program, you should place the calculator in "auto" or "exact" mode, and if the functions $p(x)$ or $q(x)$ contain any trigonometric functions, be sure to set the calculator in radian mode. It is also a good idea to perform F6: NewProb before running the program. After this is done, use F4: Define $p(x)=\cdots$ then use F4: Define $q(x)=\cdots$ (be sure to name the functions this way and use x as the independent variable). The syntax of this program is: ordinpt(x_0, $y(x_0)$, $y'(x_0)$,n), where n is the highest power of $(x-x_0)$ in the partial sum.

Example 1: Let us solve the simple equation $y''+y=0$ with initial conditions $y(0)=1$ and $y'(0)=0$ (we know the solution will be $y=\cos x$). Define $p(x)=0$ and $q(x)=1$. Obtain the "ordinpt" program from the Var-Link list. In this example, we would enter ordinpt(0,1,0,8) (we want to see up to the 8th degree term). The calculator screens are as in Figure 1:

Figure 1 – Setup for calculation and results.

Press “Enter”. Then type in “ss” (series solution) in the command line. We recognize the series as that of the cosine expanded about 0

Example 2: If we wanted the series expansion for the solution to $y''+y=0$ about the ordinary point $x_0=1$, with $y(1)=0$ and $y'(1)=1$ (up to the 6^{th} degree term), we would enter ordinpt(1,0,1,6). Figure 2 shows the result.

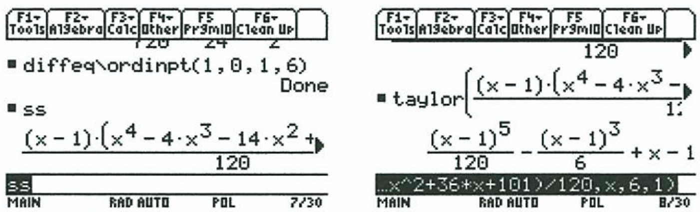

Figure 2 – Result of calculation (left) and after using the taylor command (right)

Notice that in the left screen, the calculator has expanded the binomials. If we want the series in $(x-x_0)$, use the taylor command. The syntax for the taylor command is: taylor$(f(x),x,n,x_0)$. This will give the partial sum of the taylor series for the function $f(x)$, using the variable *x* up to order *n*, expanded about x_0. The result is seen in the right screen in Figure 2.

Series Solution about a Regular Singular Point

The differential equation $y''+p(x)y'+q(x)y=0$ is said to have a regular singular point at x_0 if one (or both) of $p(x)$ and $q(x)$ fails to be analytic at x_0 but $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ are analytic at x_0. That is, $(x-x_0)p(x)=\sum_{k=0}^{\infty}p_k(x-x_0)^k$ and $(x-x_0)^2q(x)=\sum_{k=0}^{\infty}q_k(x-x_0)^k$, where each series has a positive radius of convergence. In this case, we will have at least one Frobenius series solution:

$$y=(x-x_0)^r\sum_{k=0}^{\infty}a_k(x-x_0)^k=\sum_{k=0}^{\infty}a_k(x-x_0)^{r+k},$$

where $a_0\neq 0$. Differentiating the Frobenius series twice and substituting into the differential equation, we obtain (since $a_0\neq 0$): $(r-1)r+p_0r+q_0=0$ This is called the indicial equation. Its solutions give the possible exponent of the power function that multiplies the power series in the Frobenius series. There will always be a Frobenius series corresponding to the larger of the roots of the indicial equation. If the two roots are equal or if they differ by an integer, then there may not be a second Frobenius series solution. Set $a_0=1$. Then

$$a_1 = \frac{-(p_1 r + q_1)}{r(r+1) + p_0(r+1) + q_0} \text{ and } a_k = \frac{-\sum p_{k-j} \cdot (r+j) a_j - \sum q_{k-j} \cdot a_j}{(r+k-1) \cdot (r+k) + p_0(r+k) + q_0},$$

where $k = 2, 3, 4, \cdots$ and r is the larger of the indicial roots.

The program "resingpt" performs the calculations above. The commands for this program are as follows:

```
:resingpt(xo,n)
:Prgm
:ClrIO
:xp(xo)→po:x2q(xo)→qo
:Disp "Indicial Equation & Roots:"
:Disp (r-1)*r+po*r+qo=0
:Disp solve((r-1)*r+po*r+qo=0,r)
:Pause
:Input "r=?",ro
:Local i,j,k
:0→i:0→j:0→k
:newList(n)→xplist
:For i,1,n
:(d(xp(x),x,i)|x=xo)/(i!)→xplist[i]
:EndFor
:newList(n)→x2qlist
:For i,1,n
:(d(x2q(x),x,i)|x=xo)/(i!)→x2qlist[i]
:EndFor
:newList(n)→a
:(r0*xplist[1]+x2qlist[1])/((ro+1)*ro+po*(ro+1)+qo)→a[1]
:For k,2,n
:-(xplist[k]*ro+x2qlist[k]+∑ ((xplist[k-j]*(ro+j)+x2qlist[k-j])*a[j],j,k-1))/((ro+k-
1)*(ro+k)+po*(ro+k)+qo)→a[k]
:EndFor
:(1+∑ a[k]*(x-xo)^k,k,1,n))→ss
:EndPrgm
```

To use this program, you must first define (using "F4:Define") the functions $xp(x) = (x - x_0)p(x)$ and $x2q(x) = (x - x_0)^2 q(x)$. Be sure to simplify by cancelling if at all possible. Use these names for the functions and use x as the independent variable. The syntax of the "resingpt" program is "resingpt(x_0, n), x_0 is the regular singular point and n is the highest power in the power series factor of the Frobenius series.

Example 3: Find Frobenius series solutions for the equation $2x^2 y'' - x y' + (1+x) y = 0$.

First divide both sides by $2x^2$ so that the coefficient of y''-term is 1. We get

$$y'' - \frac{1}{2x}y' + \frac{(1+x)}{2x^2}y = 0.$$

Notice that 0 is a singular point, but $x \cdot p(x) = -\frac{1}{2}$ and $x^2 \cdot q(x) = \frac{(1+x)}{2}$ are analytic at 0. Define $xp(x) = -\frac{1}{2}$ and $x2q(x) = \frac{(1+x)}{2}$. Let's look for the partial sum of the Frobenius series solution centered at the regular singular point x_0 up to the 4th degree. Figure 3 shows the setup of the calculator (left screen). Press "Enter", the program will display the indicial equation and the roots of this equation. You are prompted for the root to use. Let's use the larger root. Type in "1" and press "Enter"

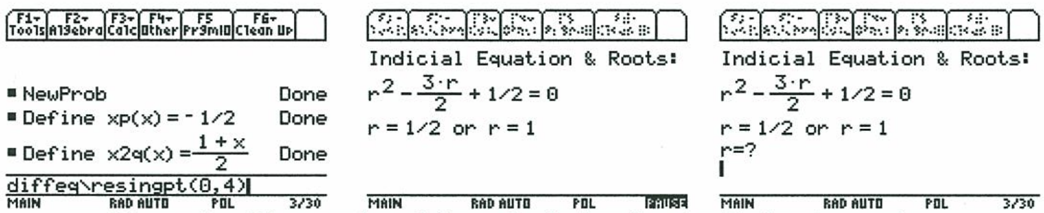

Figure 3 – The results of the calculation for the Frobenius series.

In a few seconds the calculation is done. Go to the home screen and type in "ss". The results are given in Figure 4.

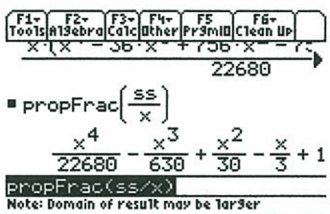

Figure 4 – The power series factor in the Frobenius series.

Thus, one Frobenius series is: $y = x^1\left(1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} + \cdots\right)$

Since the indicial roots do not differ by an integer, there will be a second linearly independent Frobenius series solution corresponding to the indicial root $r = \frac{1}{2}$. Let's find it. After running the program again (looking for the solution up to the 7th degree), selecting $r = \frac{1}{2}$, we obtain the following partial sum (you must supply the factor $x^{\frac{1}{2}} = \sqrt{x}$):

$$y = \sqrt{x}\left(1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + \frac{x^6}{7484400} - \frac{x^7}{681080400} + \cdots\right).$$