# IN SEARCH OF LOGISTIC CURVE – ACTIVITIES FOR YOUR STUDENTS FROM ALGEBRA TO CALCULUS

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## Introduction

In a traditional mathematics curriculum the logistic growth model is usually introduced to the students for the first time in an advanced calculus, when they start the topic of integration by partial fractions. Majority of students do not take calculus and thus never have an opportunity to learn about logistic model, while this is one of the most realistic models for the growth process. The following activities demonstrate that this model can be introduced to the students as early as in their algebra classes. For the students who do continue into advanced calculus, early analysis of the logistic function and its properties will lead to better understanding of the differential equation for the logistic model.

The lab activities developed to introduce logistic model utilize prototype of the Texas Instrument Corporation's latest (not yet publicly released) handheld hardware/software package called TI-Nspire™ CAS. This revolutionary technology represents not only a new generation of graphing calculator technology, but also an advance in the capabilities of a low-cost personal computing device which is reliable and easy-to-use, and which supports a broad range of instructional models and advanced modes of assessment for teaching mathematics and science. The TI-Nspire™ incorporates two new capabilities not available previously: document-based content, and the ability to display multiple representations which are connected and in a single plane. The document-based content system is an organized presentation of multiple screens of mathematics, which can be saved, shared, annotated, and revisited. The multiple representation capability dynamically links graphical curves, axes, equations and tables in simultaneous displays, such that a change in one representation is carried through to the others. All these features significantly enhance students' experience and allow capturing a progression of documents which represent stages in a student's thinking about a problem-solving task. The released product may or may not contain the same features as described below.

## Spread of Disease Simulation

This activity allows collecting data in a simulation of disease game. The goal of this game is to analyze the logistic model, which describes a restricted growth process:

$$N(t) = \frac{A}{1 + Be^{-kt}} \qquad \textbf{Equation 1}$$

In the game each student in class is assigned a number. One "infected" person is chosen at the beginning of the game. An "infected" person uses random integer program in the calculator application of TI-Nspire™ handheld or software to identify the next person to be "infected". In the precalculus version of this game the probability of the positive infection is determined by a die. Number of people infected at each roll of the die is recorded, where each roll of the die represents the time unit, for simplicity, a day. The process continues until all students get infected or the instructor stops the activity. Students analyze the scatter plot of the data, determine logistic regression equation for the data, and explore the meaning of the coefficients in the equation of the logistic function (Equation 1). Sample data for the class of 20 students are shown on Figure 1.

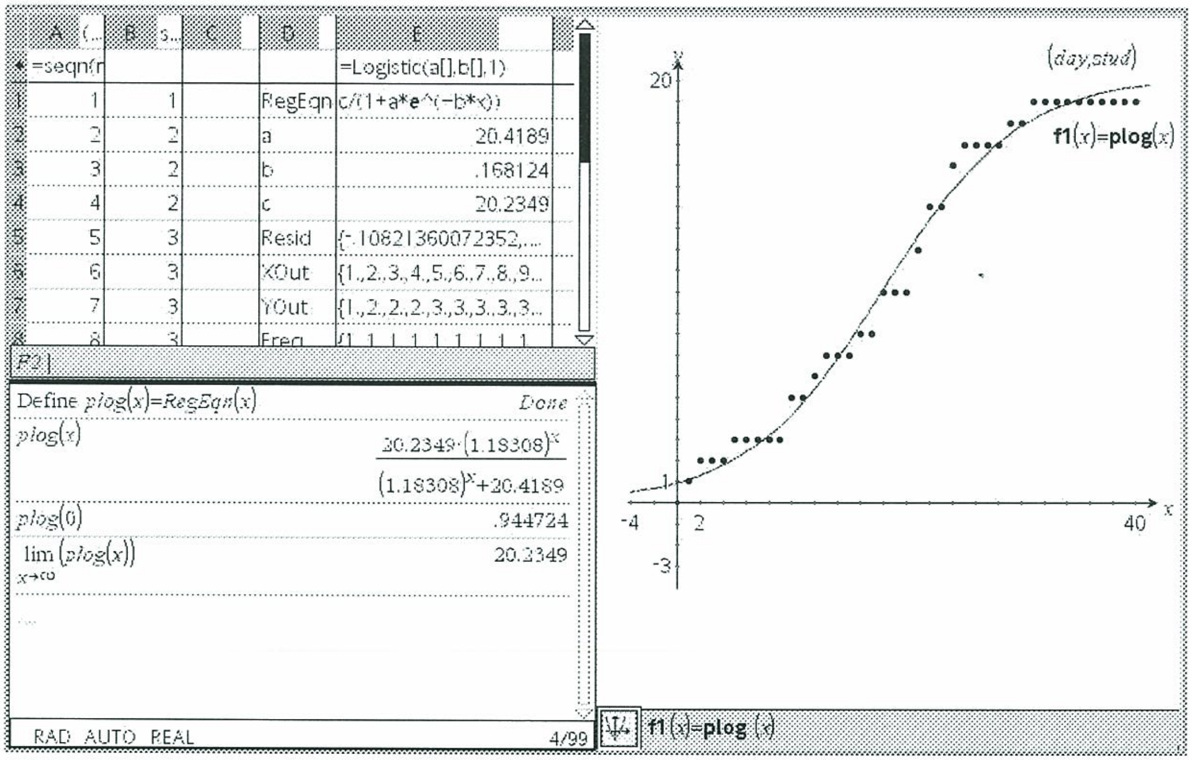


**Figure 1. Spread of Disease Simulation – Sample Data for Precalculus Version**

In this experiment using logistic regression on the data yields: $y = \dfrac{20.23}{1+20.4e^{-.168x}}$. The horizontal asymptote of this function is $y$ = 20.23. The percent difference between this number and population size is 1.2%. Initially at $x = 0$, $y = 20.23/21.4 = 0.94 \approx 1$ person.

This activity allows students to evaluate restricted growth process and to understand the shape of the logistic function based on the experiment. Since the number of possible infected people is restricted, students have a very good conceptual understanding of the range of the function. With the assumption that no one is treated for the infection, the monotonic increasing behavior of the function is understood well by the students. The concavity of the function requires additional discussion and may involve analysis of the rate of change of the function.

In calculus version of the activity the probability of positive infection is assumed to be 100%. Students use random integer generator in the calculator application of TI-Nspire™ handheld or software in order to determine the next infected person. The process continues for the fixed number of "days", but repeated several times for better statistics. Sample scatter plot of data along with logistic regression are shown on the Figure 2.

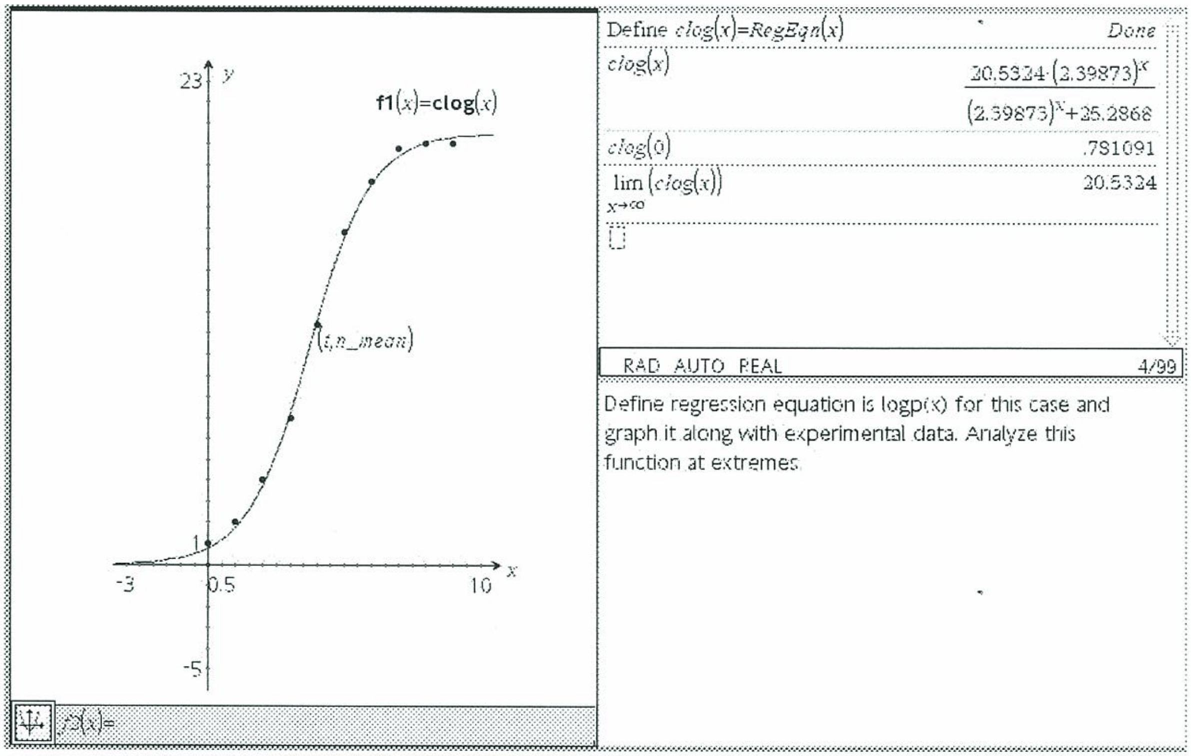


**Figure 2. Sample Data and Logistic Regression Equation - Calculus Version**

Students use experimental data to investigate behavior of the growth rate of this function, determine experimentally differential equation for the logistic model, solve differential equation with the given initial condition and compare this solution with the regression equation. TI-Nspire™ allows students to use CAS to verify their work.

The data in both versions of the activity provide logistic shape function. This simulation game activity is developed for the TI-Nspire™ CAS from the earlier activities developed by the author (Lyublinskaya, 2003a & b).

How Fair Is the Drug Test

In this activity students analyze the conditional probability of false-positive results in a drug testing procedure. The social key issues, such as non-medical use of anabolic/androgenic steroids among adolescents and young adults, mandatory drug testing in college athletics, and cost of the drug tests make this activity relevant for the students. At the same time students are engaged in rich mathematical content ranging from conditional probability, through iterations and logistic growth model, to differential equations.

In the beginning of this activity students use TI-Nspire™ Spreadsheet application to explore the effect of different factors on the probability of false-positive results. These factors are the size of the tested population, the fraction of actual drug users in this group, and the accuracy of the drug test. Due to dynamic nature of TI-Nspire™ they are able to vary values of the parameters and observe their effect on the conditional probability of false-positive results. The results are astonishing! Using 99% accuracy test with the tested group that has only 3% of actual drug users, the probability of false-positive results is 25%. With 95% accuracy it grows to 63%! Students determine that the repetitive testing could help reducing the probability of false-positive results. This leads to the iterations of two functions, false-positive probability and fraction of actual drug users in the test group that produces logistic function, which can be received as regression curve and as closed form for the iterations (Figure 3).

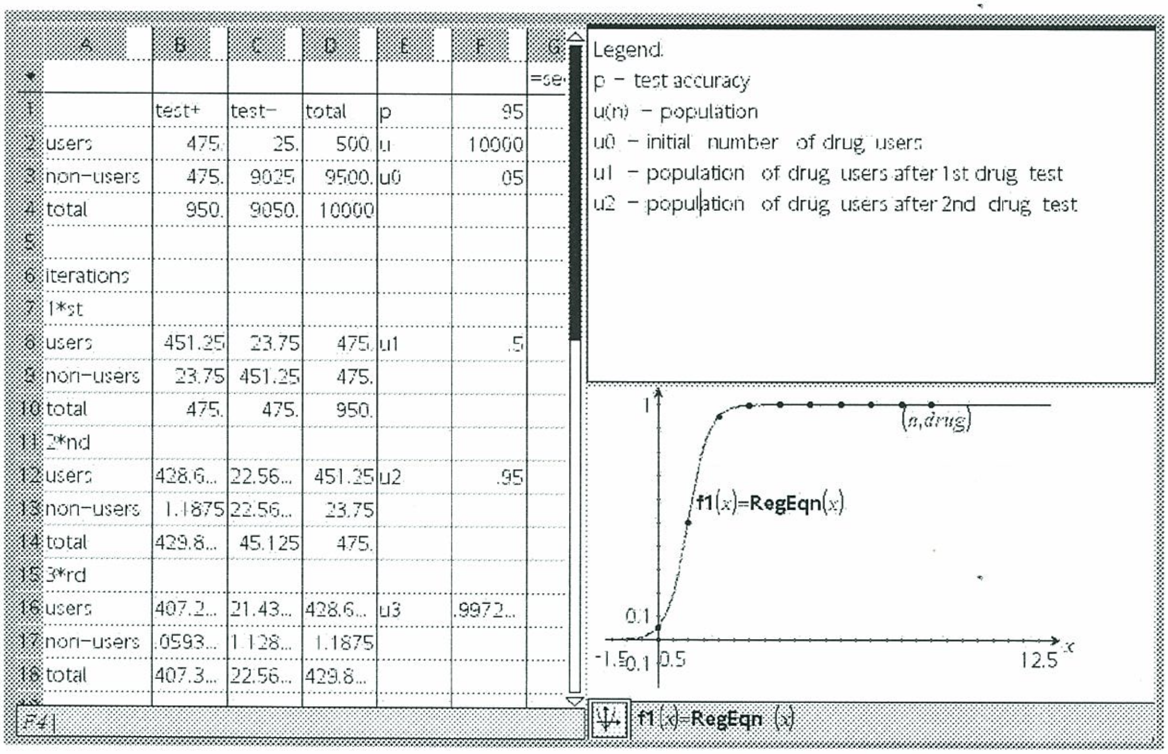


**Figure 3. Analysis of Iterations and Regression for Drug Testing**

Students analyze the recursive formula for the fraction of drug users and derive closed form equation. Let $u_0$ be the initial fraction of drug users in the sample group. Then, $1-u_0$ is the fraction of non-users. Let $p$ be the accuracy of the test. The process of iteration can be written with the help of a recursive formula for the fraction of drug users, $u_n$, after the $n^{\text{th}}$ test as follows:

$$u_n = \frac{pu_{n-1}}{pu_{n-1} + (1-p)(1-u_{n-1})} = \frac{u_{n-1}}{(1-r)u_{n-1} + r}$$ **Equation 2**

where $r = \frac{1}{p} - 1$.

The closed form for this recursive formula is easy to find algebraically or using TI-Nspire™ CAS capabilities. The closed form equation is identical to the regression equation for the logistic model, which confirms the fact that the iteration leads to the logistic function.

Considering iterations further, it is possible to show that this process leads to the differential equation for the logistic function. Details of mathematics for this problem could be found in the earlier work published by the author (Lyublinskaya, 2005).

## Conclusion

In the TI-Nspire™ environment, instructor could provide students with the document that includes tasks and questions for them to follow or ask students to start with the blank document. The document could have several problems each consisting of one or more pages. Within each problem defined variables, lists, etc. preserve their definitions while students move from one page of the problem to another. Students could enter data in spreadsheets/lists, link these data to the graphs, perform spreadsheet operations using formulas and linking different cells, find regression equations for different sets of data, use calculator CAS to solve differential equation, etc. The multiple representations used by the students in the process of modeling and problem solving are linked interactively, which allows them to alter various conditions and observe changes in the graphs, functions, etc. All work done by the students is saved in the document, which allows the instructor to follow students' thought process and see their work step by step.

As a result of using these activities students can recognize the shape of the graph of the logistic function. They learn that the meaning of the coefficient $A$ in the equation for the logistic function (Equation 1) is the population size and that the meaning of the coefficient $B$ is the initial ratio of infected to the non-infected population: $B = \frac{u_0}{A - u_0}$ (here $u_0$ is the initial number of infected people). Students learn that the coefficient $k$ in the logistic function is related to the ratio of geometric series in the iterations process, $r$. They analyze the behavior of the rate of change and determine that it reaches the maximum at half population, $A/2$, derive and solve differential equation for the logistic function.


## References:

Lyublinskaya, I. (2003a) Connecting Mathematics with Science: Experiments for Precalculus. Emeryville, CA: Key Curriculum Press.

Lyublinskaya, I. (2003b) Connecting Mathematics and Science: Experiments for Calculus. Emeryville, CA: Key Curriculum Press.

Lyublinskaya, I. (2005) How Fair Is The Drug Test? *The Mathematics Teacher, 98*(8), 536-543.