INVESTIGATING THEOREMS IN MATHEMATICS AND STATISTICS USING TECHNOLOGY

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Abstract
In this paper, we will discuss the use of Computer Algebra Systems (CAS) and Statistical software to investigate various theorems in Mathematics and Statistics. Specifically, we'll present examples to demonstrate the use of Maple and Geometry Exploration Software (GES) to validate the expected results of theorems in calculus and geometry.
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Introduction
In the past two decades, technology-based instruction using tools such as graphing calculators, computer algebra systems, statistical packages and Geometry Exploration software has become an essential component of undergraduate mathematics. At Texas Lutheran University we introduced graphing calculators to our classes about twenty year ago. During the early 90's we began using Maple in our mathematics classes, making us one of the first schools to use this technology. We have also utilized statistical packages, such as Minitab, in our statistics courses. Recently we have begun using the Geometry Exploration software with students majoring in education. It has been our experience that simple presentation of a theorem, even proof of a theorem, does not satisfy the curiosity of most students. They must see the application and verification of a theorem to appreciate its significance. The traditional paper and pencil tools are cumbersome and impractical when used to generate the multiple instances necessary to establish the plausibility of theorems. Current mathematics technology creates an environment for a more practical exploration process and a more enjoyable learning experience. In this paper, we have attempted to share some our experiences in using technology to explore various theorems. For the sake of brevity, we have limited our examples to the theorems related to topics in geometry, calculus and statistics. For examples related to post-calculus topics and examples demonstrating the shortcomings of technology (specifically Maple) we refer the reader to references at the end of this paper.

In the following pages we'll present three examples to demonstrate our work. For a complete list of the projects, the reader is encouraged to contact the authors.

1 Exploring the Fundamental Theorem of Calculus

In most calculus textbooks, the Fundamental Theorem of Calculus is introduced in two parts. One is the practical part, which allows a student to determine the value of a definite integral by evaluating an antiderivative at the upper and lower bounds. The second part simply states that if F(x) is defined as the integral of f(t) for a ≤t ≤x, then (assuming f is continuous), F′ = f. To illustrate the theorem, most of the time a student chooses a simple function f(t), and using the first of the theorem verifies the second part. However, this process of verification fails if f(t) is a more complicated function. In the following example, we'll use of Maple to numerically evaluate the antiderivative of g(t) = exp(-x^2) at selected points and then numerically differentiate the result. Finally, we'll use Maple to graph of the resulting values and show that it perfectly matches the graph of f(t).

```
> g:=x->exp(-x^2);
```

$$g := x \to e^{(-x^2)}$$

```
> with(student):
> k:=10;ptsGprime:=array(1..k);inc:=0;Gvalold:=0;
```

$$k := 10$$

$$ptsGprime := \text{array}(1 .. 10, [\ ])$$

$$inc := 0$$

$$Gvalold := 0$$

```
> for i from 1 to 10 do
inc:=inc+0.1;Gvalnew:=evalf(simpson(g(x),x=0..inc,16));Gprime:=(Gvaln
ew-Gvalold)/0.1;ptsGprime[i]:=[inc-0.05,Gprime];Gvalold:=Gvalnew; end
do;
```

$$inc := 0.1$$

$$Gvalnew := 0.09966766429$$

$$Gprime := 0.9966766429$$

$$ptsGprime_1 := [0.05, 0.9966766429]$$

$$Gvalold := 0.09966766429$$

.

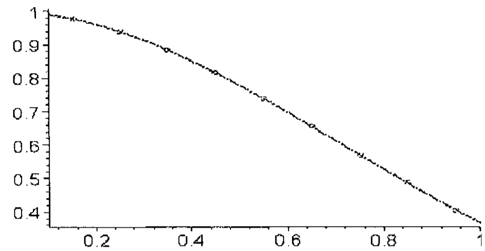
.

We have omitted the rest of the output for the sake of brevity.

```
>plotGprime:=[seq(ptsGprime[i],i=1..k)];
```

$$plotGprime := [[0.05, 0.9966766429], [0.15, 0.9769736701], [0.25, 0.938728534], [0.35, 0.884149629], [0.45, 0.816281794], [0.55, 0.738725419], [0.65, 0.655321699], [0.75, 0.569842184], [0.85, 0.485716807], [0.95, 0.405826196]]$$

```
> graph1:=plot(plotGprime,style=point,symbol=circle,thickness=5):
> graph2:=plot(g(x),x=0.1..1,thickness=2):
> plots[display]([graph1,graph2]);
```



As expected the graph of the (numerical) values of the derivative of (the numerical) integral of f(t) overlaps the graph of f(t), thus verifying the statement of the theorem.

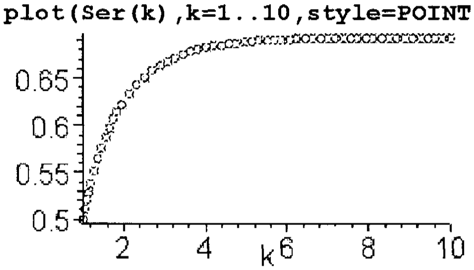
2 Exploring a Remainder Theorem for Series with positive terms

A portion of this Theorem [3] states the following: Suppose that Σa_n is a converging series, B= limit of $r_n = a_{n+1}/a_n$ as n approaches ∞ and $\{r_n\}$ is an increasing sequence. Then the remainder after n terms, R_n, is such that $R_n \le a_{n+1}/(1\text{-}B)$. Testing such a theorem is a very difficult task for a typical student. For example, finding the limit of most series (except for Geometric Series), even if they exist, is beyond the reach of most students. In the following example we'll investigate this theorem using a problem taken from [3].

`> a:=n->1/(n*2^n) ;` $a := n \to \frac{1}{n\, 2^n}$

`> Ser:=k->sum(1/(n*2^n),n=1..k) ;` $Ser := k \to \sum_{n=1}^{k} \frac{1}{n\, 2^n}$

`> plot(Ser(k),k=1..10,style=POINT,symbol=CIRCLE) ;`



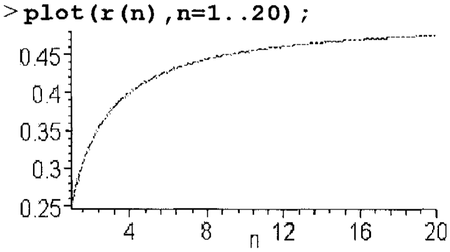
The plot indicates convergence, but let's use the ratio test to prove it.

`> r:=n->a(n+1)/a(n) ;` $r := n \to \frac{\mathrm{a}(n+1)}{\mathrm{a}(n)}$

`> B:=limit(r(n),n=infinity) ;` $B := \frac{1}{2}$

Plot r versus n to show that $\{r_n\}$ is an increasing sequence:

`>plot(r(n),n=1..20) ;`



Now we'll use Maple to find the limit of the series, S, sum of the first seven terms, s7, the actual remainder $|S - s_7|$ and an upper bound for the remainder.

```
> limit(Ser(k),k=infinity);
```

$$\lim_{k \to \infty} \ln(2) - \frac{2\,\mathrm{LerchPhi}\left(\frac{1}{2}, 1, k\right)}{2^{(k+1)}} + \frac{1}{2^k\,k}$$

```
> S:=evalf(%);
```
$S := 0.6931471806$

```
> s7:=evalf(Ser(7));
```
$s7 := 0.6922619048$

```
> ExactR:=abs(S-s7);
```
$ExactR := 0.0008852758$

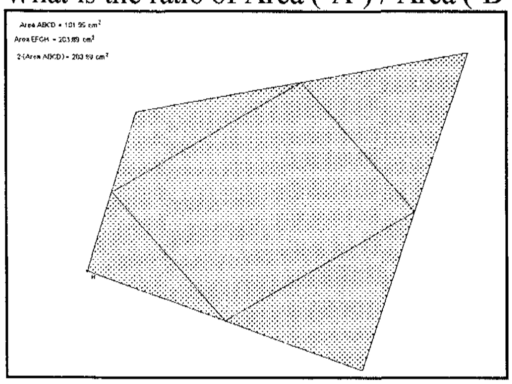
```
> upperR:=evalf(a(8)/(1-B));
```
$upperR := 0.0009765625000$

The result is consistent with the Theorem, which clearly gives us an upper bound for the remainder.

3 Exploring Geometry with GES

In our math course for future teachers the students generally underestimate their ability to understand and do mathematics. This is an example of one simple textbook problem that, with the help of GES led to some interesting mathematics.

Our text book asked that we create a quadrilateral 'A' and then find and connect the midpoints of the sides of 'A,' thus creating a new quadrilateral 'B' that lies inside 'A.' What is the ratio of Area ('A') / Area ('B')?



Using GES the students were able to draw the appropriate figure and calculate the value of the ratio. They found Area ('A') / Area ('B') = 2.0.

By distorting the original figure the students suddenly had multiple examples, all of which suggested that the ratio Area ('A') / Area ('B') was always two, independent of the quadrilateral with which they started.

Because of the ease of using GES the students will ask the next question: namely, if we used triangles instead of quadrilaterals, when we compare areas, would we get a ratio of two?

With their first triangle they found Area ('A') / Area ('B') = 4 and by distorting the figure they found empirical evidence that for all triangles, Area ('A') / Area ('B') = 4. Quickly they moved on to figures with more sides and discovered (empirically) the following:

sides	ratio
3	4:1
4	2:1
5	1.53:1
6	1.33:1

This lesson presents a great opportunity to discuss the distinction between empirical and deductive results. The students know that examples suggest but do not prove results. So, we have formulated a loose hypothesis that if figure 'A' is an n-gon and figure 'B' is an n-gon constructed by connecting the mid points of the sides of 'A' that the ratio Area ('A') to Area ('B') is constant. To **prove** this result we need to use deductive logic. To simplify the proof we amend the hypothesis to include only regular polygons. Without too much difficulty they find (or at least agree) that the area of a given original polygon is given, as a function of the number of sides, n, and the radius of a circumscribed circle, r, by

$$n * r^2 * \sin(\frac{n-2}{2} * 90) \cos(\frac{n-2}{2} * 90)$$

And the area of the inner polygon created by connecting mid-points of the original figure

Is given by, $$n * r^2 * \sin^3(\frac{n-2}{2} * 90) \cos(\frac{n-2}{2} * 90)$$

Taking the ratio Area ('A') / Area ('B') yields Area ('A') / Area ('B') $= \frac{1}{\sin^2(\frac{(n-2)*90}{n})}$

This is a formula that we have created through deductive reasoning. It can easily be compared to our empirical results:

# sides	deductive	emperical
3	4.0000	4.0
4	2.0000	2.0
5	1.5279	1.53
6	1.3333	1.33
7	1.2319	
8	1.1716	
9	1.1325	
10	1.1056	

An important result will follow when the students point out that the above deductive result was arrived at using regular polygons. Do the results hold for arbitrary n-gons? This leads to a discussion of the role of counter examples in deductive logic. With a bit of encouragement the student can construct an example that proves that the above result does not hold for arbitrary polygons.

Conclusion: Technology can be a powerful tool for motivating our students and for enabling them, with guidance, to investigate mathematical results that in an earlier time would have been beyond their reach. The software and hardware does not substitute for deductive logic and proof, but it can motivate the student and support our claims concerning the necessity of formal proof.

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