PRECALCULUS, IT'S OUT OF THIS WORLD

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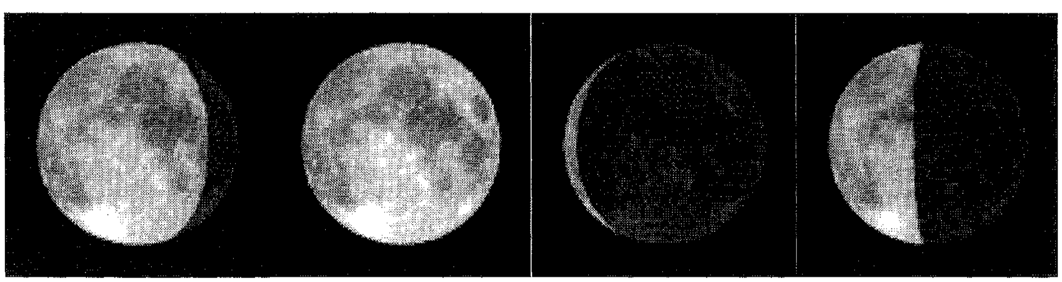
Teaching precalculus using a contextual thematic approach can breathe new life into the course. Generally, I select two or three major contexts so that students see variety and I revisit each major context multiple times throughout the semester to bring depth to the discourse. One of my favorite contextual themes is astronomy.

This paper contains two of the examples presented at the ICTCM conference (2006). These examples are drawn from labs in *Precalculus: Concepts in Context* [1], a combined text and labs-projects-explorations manual.

Labs are designed for group collaboration, deal with a single context and problem(s) related to that context, and require use of a graphing calculator. Labs culminate in a single written report submitted by the group. Requiring a single report from the group changes the dynamics of how students work within a group. Since all group members have a vested interest in the outcome, they actively discuss the problems posed in the lab and critique each others ideas, solutions, and writing. To hold individual group members accountable, I include questions on exams and quizzes based on the assigned labs and other group work.

Modeling Moonlight

If you look at the moon on a series of clear nights, you'll notice that the size of the bright portion changes from night to night. The photos in Figure 1 display the moon's appearance on four nights in March 2006. (This time period was selected to coincide with dates of the 2006 ICTCM conference.)



March 17 March 20 March 23 March 26

Figure 1. The Moon Around Time of 2006 ICTM Conference

Before students begin work on this lab, I ask them whether or not they think that the illuminated fraction of the moon's visible disk changes at a constant rate from day to day. Students are generally clueless and hence, after a short discussion, students are ready to consider some data. Figure 2 contains three-months worth of data on the illuminated fraction of the moon's surface visible from the earth. It turns out that 1999, the year these data were collected, was a rather special year. There were two blue moons (the second full moon in a single month) during that year.

Date	Day	Illuminated Fraction
1/1	1	0.99
1/7	7	0.73
1/13	13	0.19
1/19	19	0.03
1/25	25	0.55
1/31	31	1.00
2/6	37	0.71
2/12	43	0.17
2/18	49	0.05
2/24	55	0.62
3/2	61	1.00
3/8	67	0.70
3/14	73	0.15
3/20	79	0.08
3/26	85	0.69

Figure 2. Fraction of the Moon Illuminated at Midnight (E.S.T.) by Date in 1999.

To get a visual impression of how the moon's illuminated fraction is changing, students make scatterplots of the illuminated fraction versus the day. Most students use ZoomData to automatically select a viewing window (see Figure 3, left-hand scatterplot). However, in this window students generally think the dots are randomly scattered. The periodic pattern is more evident after widening the (ymin, ymax)-interval (see Figure 3, right-hand scatterplot).

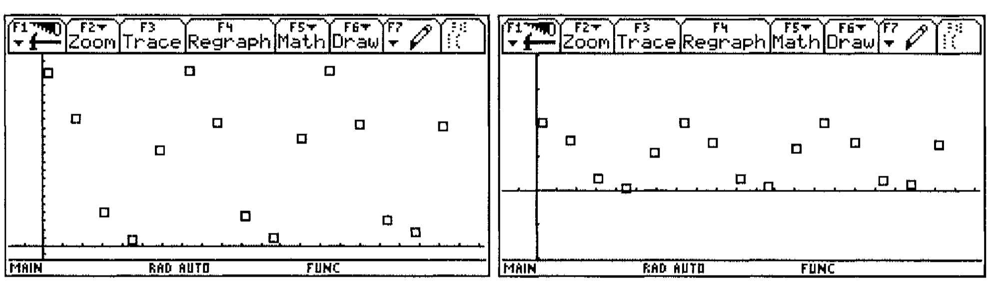


Figure 3. Scatterplots of the Moonlight Data Using Different Window Settings.

Since the pattern in the right-hand scatterplot resembles a cosine wave, students first develop a model of the form $y = A\cos(B(t - C)) + D$. Here's an approach one group used and their final model:

- Since the moon oscillates between new moon (totally dark) and full moon (totally illuminated), the values for A and D should be 0.5.
- The moon is full on days 31 and 61 and hence, $B = 2\pi/30$.
- Based on the additional information that the moon was full on January 2, shift the graph two units to the right by setting $C = 2$.
- After graphing the preliminary model and the data in the same window, a few adjustments were made to improve the fit. Final model:

$$M(t) = 0.5\cos[(2\pi/29.7)(t-2)] + 0.5\,.$$

Using this model, students analyze rates of change by evaluating the symmetric difference quotient

$$\frac{M(t+0.1) - M(t-0.1)}{0.2}$$

for various values of t. They discover that this rate of change is largest at times when half the moon is illuminated and zero at times of the new moon or full moon. In other words, if on one night the moon appears full, it will appear nearly full the next night. However, if on one night the moon is half illuminated (first or third quarter), the next night it will appear noticeably different. After completing this lab, students often tell me that they now make it a habit to look at the moon when they are out at night. They seem amazed that the mathematics works!

Modeling Outer-Planet Light

Moving from the moon farther out into the solar system, we come to Mars, Jupiter, and Saturn, the three planets which are visible from the earth with the naked eye and whose orbits lie beyond that of the earth. The key question is: Do Mars, Jupiter and Saturn have phases similar to those of our Moon?

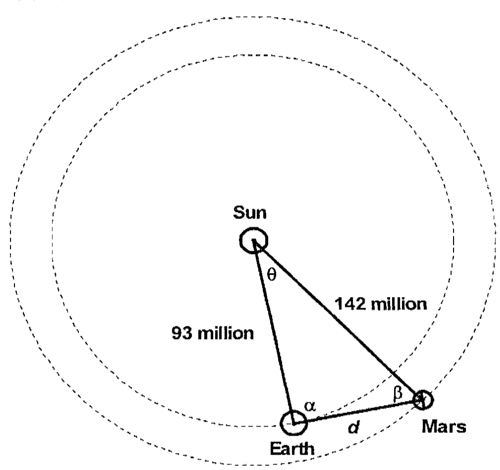


Figure 4. Earth and Mars Orbiting Sun, View From North Star (units in miles)

For the purposes of this paper, discussion is restricted to Mars. Figure 4 shows Earth and Mars orbiting the sun as seen from the North Star. For our model, we assume orbits are circular. Only one hemisphere of Mars is visible from the earth. (Similar to the moon, it appears as a disk.) In addition, one hemisphere of Mars is illuminated by the sun. In this lab, students investigate how much of the illuminated half of the planet is visible from earth and whether that portion remains constant or changes over time.

Figure 5 shows a magnified view of Mars that has been rotated so that the earth is positioned at the bottom. A diameter has been drawn through Mars perpendicular to the line connecting Mars to the earth (Mars' earth-diameter). Another diameter has been drawn through Mars perpendicular to the line connecting Mars to the sun (Mars' sun-diameter).

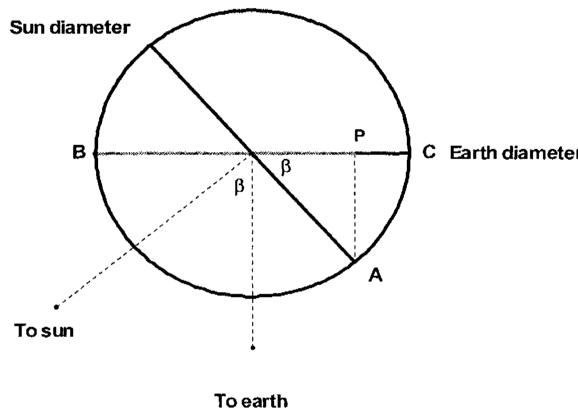


Figure 5. Magnified View of Mars

Students' task in this lab is to determine the maximum and minimum fractions of Mars' earth-diameter that could be illuminated by the sun. The idea is this: If the entire earth-diameter is illuminated, then the entire planet, viewed from the earth, appears bright; if a smaller amount of the earth-diameter is illuminated, then only a portion of Mars' surface appears bright to us.

Notice that when $\beta = 0$, Mars' earth-diameter is fully illuminated. However as β increases, the illuminated fraction of Mars' earth-diameter is decreases. (See Figure 5.) Notice also that β depends on α (Refer to Figure 4.). Hence, we seek a relationship between β, the Sun-Mars-Earth angle, and α, the Sun-Earth-Mars angle. Using the Law of Sines, we can express β in terms of α:

$$\beta = f(\alpha) = \sin^{-1}\left[\frac{93\sin(\alpha)}{142}\right]$$

Notice that $\beta = 0$, its minimum value, when $\alpha = 0°$ and $\alpha = 180°$ (Ignore possible eclipses.). Figure 6 shows the graph $\beta = f(\alpha)$. From this graph we find that the maximum value for β is approximately $41°$.

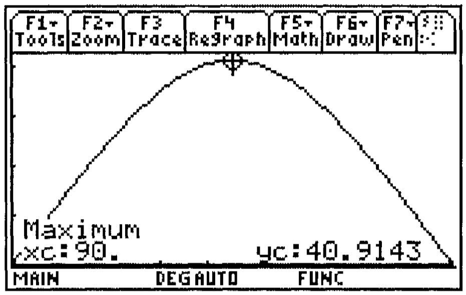


Figure 6. Graph of $\beta = f(\alpha)$.

All that is left is to determine what fraction of Mars' earth-diameter is illuminated when $\beta = 41°$. Mars' diameter is approximately 4,220 miles. The length of the illuminated portion is *BP* (See Figure 5.). The illuminated fraction corresponding to $\beta = 41°$ is:

$$\frac{2110 + 2110\cos(41°)}{4220} \approx 0.88$$

Hence, Mars does have phases. It oscillates from 88% illuminated to fully illuminated.

Other "Out-Of-This-World Topics

There are many other topics related to astronomy that require only precalculus level mathematics. Here are three more examples which can be found in *Precalculus: Concepts in Context* [1]:

- The gravitational attraction on a spaceship between the earth and the moon is a function of the reciprocal of the square of its distance from the earth and the reciprocal of the square of its distance from the moon.
- Tycho Brahe's geocentric model of the solar system can be modeled using parametric equations.
- For conic sections consider the eccentricity of planets' elliptical orbits and the hyberbolic path of an asteroid deflected from its elliptical orbit about the sun by the earth's gravity.

Give "out-of-this-world" topics a try. Both you and your students will have some fun. Students will see the relevance of mathematics both in their world and out into the solar system.

Reference

[1] Moran, J., Davis, M., and Murphy, M. *Precalculus: Concepts in Context 2e* Brooks/Cole Publishing Company, 2004.