VISUALIZATIONS OF VECTORS, VECTOR-VALUED FUNCTIONS, VECTOR FIELDS, AND LINE INTEGRALS

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For over seven years we have been teaching the calculus courses of the IMPULSE program designed for engineering and physics students. We have observed that many students have difficulty in visualizing basic operations on vectors including sum, difference, scalar multiplication and normalization as well as the dot product of two vectors, projection of a vector onto a second vector, and the cross product of two vectors. Moreover, although our students eventually memorize the steps necessary to evaluate a line integral for a calculus exam, many of our students have difficulty in internalizing the concept of the line integral. Consequently, all too often our students have difficulty with the line integrals encountered in a physics or engineering context (perhaps involving some change in notation). To help our students internalize the concepts of vectors, vector fields and line integrals we have added a family of vector visualization tools to a new software package that we are developing called *The Image Data Modeler*. This software package

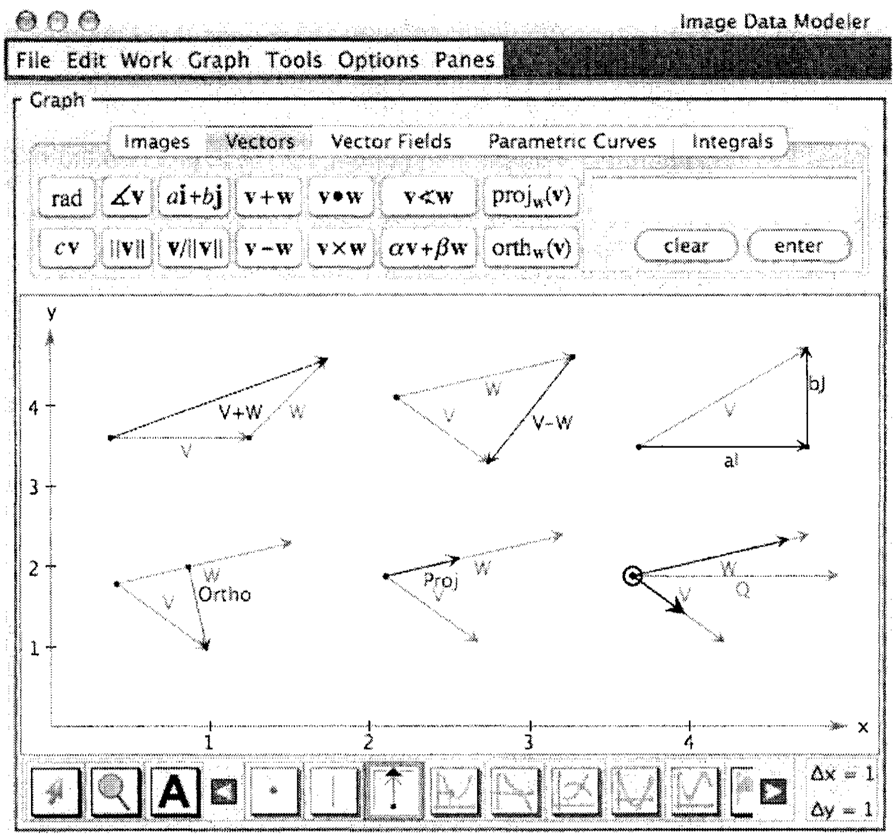

Figure 1 The Vector Interface

is a new version of our Mac only package called *TEMATH* (see below). This new package runs on computers with Macintosh OS X, Linux, or current versions of Microsoft Windows/NT/XP operating systems that have a Java JVM 1.4.2 (or later) installed. Among its many features, our new package has the capability of plotting vectors, 2D vector fields, parametric paths including their tangent vectors as well as evaluating line integrals. Moreover, a digital image of some physical phenomenon can be imported and used to generate a set of data for modeling purposes.

Vectors

The vector interface (see Figure 1) allows a user to enter a vector using the click and drag method or, more exactly, by entering the coordinates of the endpoints. Additionally, we've implemented the following vector tools: scalar multiple, norm (length), angle, <i, j> components, unit vector, sum of vectors, difference of vectors, dot product, cross-product (2D), angle between vectors, linear combination of vectors, projection of one vector onto another vector, and orthogonal component of one vector with respect to another vector. To help students visualize these vector operations, we have included some animations. For example, when two vectors are added, the second vector is slowly moved to align its tail with the head of the first vector, and then the sum vector is displayed. Vectors can be drawn in different colors and thicknesses.

A typical applied problem using vectors in our IMPULSE calculus course is to determine the tension in two ropes supporting a mass. To make this problem come to life for students, you can take a digital picture of two of your students holding the ropes and mass (see Figure 2). Ask your students to use their intuition and determine which of the two rope holders is holding more of the weight. This image can then be imported into the vector interface. Students can draw the three vectors needed to solve the problem and use the linear combination tool to find the tensions along the two ropes. If you can borrow some force measuring instruments from a Physics Department, students can then measure the tensions along the ropes and compare them with the theoretical results. This is a fun way for them to check their hand-calculated solution. Note that in Figure 2 we pasted enlarged images of the measured force scales for a quick reference when performing these calculations.

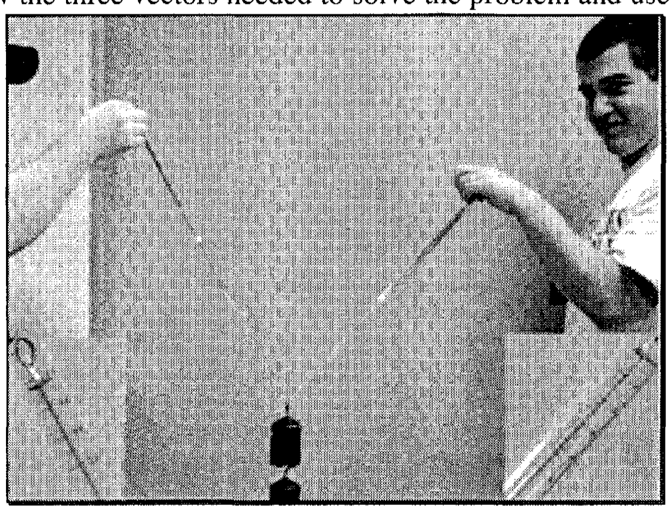

Figure 2 Sharing the Load

Another vector example that can be used with calculus students is to analyze the collision of two hockey pucks on an air table. Again, with the help of the Physics Department, we made a video of the collision of two hockey pucks and imported the frames of the video into *The Image Data Modeler*. Using the Point and Vector tools, students can find the state vector before and after the collision. In theory, assuming a perfectly elastic

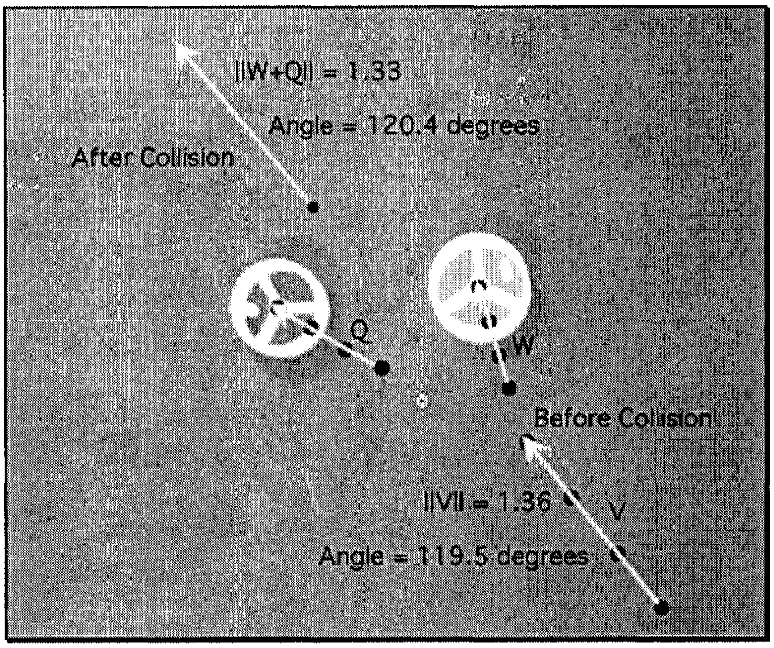

Figure 3 Collision of Two Hockey Pucks on an Air Table

collision, these vectors should be the same. Our experimental results shown in Figure 3 are very close.

Vector Fields

We have included a number of the standard vector fields used in a second year calculus or physics course, for example,

$$\mathbf{F}(x,y) = y\mathbf{i} - x\mathbf{j},$$

$$\mathbf{F}(x,y) = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}},$$

and the *area vector field*

$$\mathbf{F}(x,y) = -0.5y\mathbf{i} + 0.5x\mathbf{j}.$$

The user has the option of changing the length, color, thickness, and number of the vector field arrows. An example vector field is shown in Figure 4.

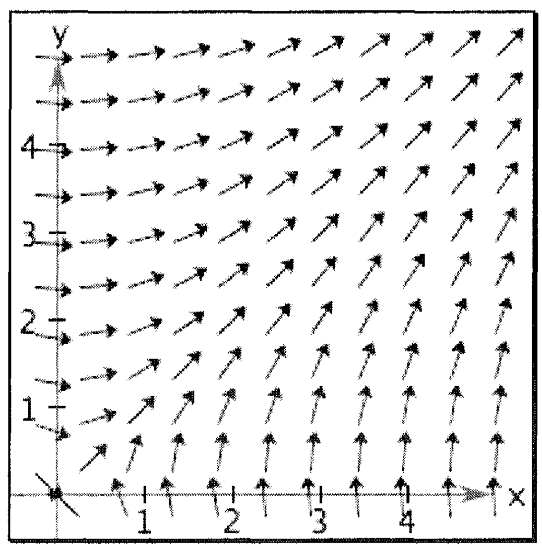

Figure 4 $\mathbf{F}(x,y) = (y\mathbf{i} + x\mathbf{j}) / \| < x,y > \|$

Line Integrals

The following paths for line integrals are included in the software: line, rectangle, arc, triangle, polyline (line segments connecting data points forming a polygon), and a general path (composed of a combination of the above paths). Our line integral interface (see Figure 5) allows students to step along the path in small increments. As they step along, the tangent and field vectors at the point are drawn allowing the student to visualize the contribution of each step to the integral. They can easily visualize when there is a positive or negative contribution. In addition to stepping (both forward or backward), students can click the Animation button and watch the movement along the entire path or, simply, click the Calculate button to find the value of the line integral. With these tools, students can easily investigate the concepts of path dependence/independence, conservative fields, circulation, Green's Theorem, and more.

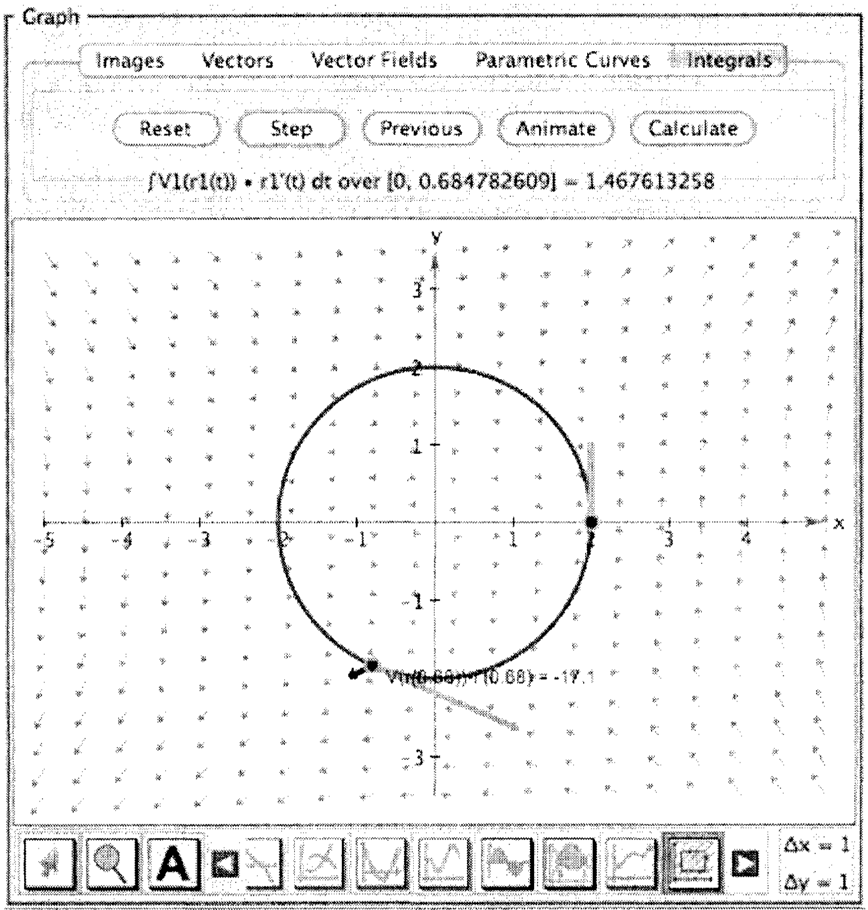

Figure 5 The Line Integral Interface

For example, to demonstrate the usefulness of Green's Theorem, students can approximate the area of a pond P by importing a digital photo of the pond and using *The Image Data Modeler*'s fitting tools to fit a piecewise smooth curve C_P to its boundary ∂P. Then by using the package's Line Integral tool, the area of the pond can be approximated by evaluating the line integral

$$\frac{1}{2}\int_{C_P}(xdy - ydx) \approx \frac{1}{2}\int_{\partial P}(xdy - ydx) \overset{\text{Green's Theorem}}{=} \int_P 1\, dA = \text{Area of pond}\,.$$

Figure 6 shows the use of Green's theorem to calculate the land area of the state of Florida using a very rough digital map of Florida. Using this map and Green's theorem, we calculated the area to be 55,006 square miles. A good approximation to the 53,927 square mile area of Florida indeed! In this example, care needs to be taken to set the axes lengths to match the scale of the map.

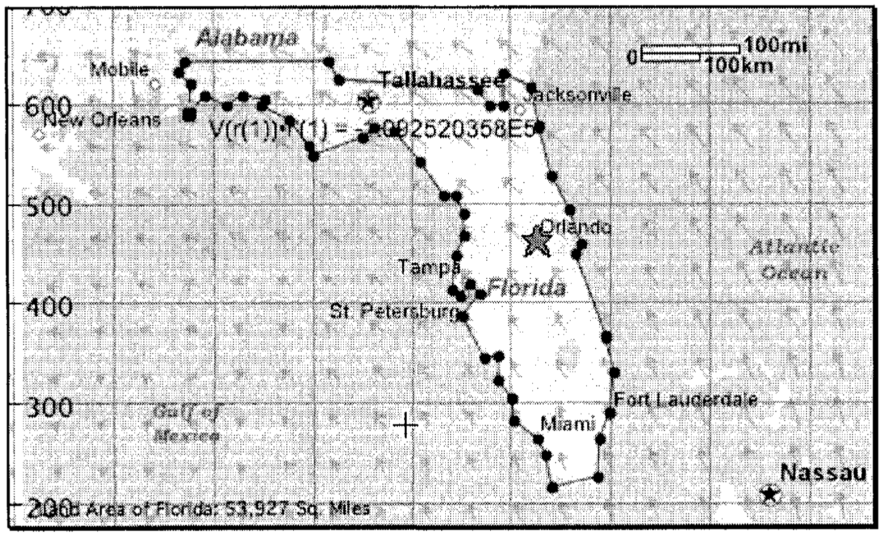

Figure 6 Using Green's Theorem to Find the Land Area of Florida

TEMATH (Tools for Exploring Mathematics)

TEMATH (Tools for Exploring Mathematics) is a mathematics exploration environment useful for investigating a broad range of mathematical problems. It is effective for solving problems in pre-calculus, calculus, differential equations, linear algebra, numerical analysis, and math modeling. TEMATH contains a powerful grapher, a matrix calculator, an expression calculator, a differential equation solver, a facility for handling and manipulating data, numerical mathematical tools, visual and dynamic exploration tools, and the capability for importing digital background images. TEMATH requires an Apple Macintosh computer running MAC OS 8.5-9.2.2 or Mac OS X (as a classic application), a 12" or larger monitor screen, 3 MB of free RAM, and 2MB of disk space (for TEMATH plus its support files). You can download a copy of TEMATH and its documentation, application files, and games from:

www2.umassd.edu/TEMATH

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