Using Technology to Teach Mathematics Topics Foundational to Calculus at the Secondary Level

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Abstract. In the last decade the integration of technology in the teaching and learning of mathematics has increased substantially. As a result many graphical ideas, some numerical ones, and some good modeling problems have found their way into Precalculus books. At the same time, relevant concepts, tools, and approaches, now available via technology, are being ignored. In this article we review examples of approaches and applications, not yet present in most Precalculus books that illustrate how the numerical and graphical capabilities of graphing calculators can be used to enhance the teaching and learning of key calculus concepts.

Prior to the integration of technology, the accessibility of various topics in secondary was fairly linear. Problems were categorized as belonging to algebra I, algebra II, geometry, precalculus or calculus. Thus, optimization problems were first studied in calculus since they typically required trigonometry and differential calculus. The introduction to functions and their properties was mainly analytical, graphs were better addressed in calculus, and very little was done numerically. Topics studied were, in part, limited by the constrains imposed by the extent of the calculations needed and time they required. In addition, mastering calculations was essential to finish problems. Hence, the teaching of algorithms often took precedence over conceptual understanding and the study of relevant applications. Moreover, the time required for exploration and discovery without the use of technology did not contribute to make these regular classroom activities. Finally, teaching tended to be more teacher-centered.

After the integration of technology started, the distinction between activities appropriate to students at various courses and ability levels becomes less clear. The ability to bridge over cumbersome calculations via technology allows students at various levels to i) use technology to meaningfully explore concepts and problems previously proposed to the most advanced mathematics students, and to ii) extend the breadth and depth treatment of these concepts. Technology facilitates changing the focus to a more conceptual one, while including more relevant applications, and increasing classroom exploration and discovery. Teaching is becoming more student-centered, with inquiry playing an increasingly bigger role on the delivery and on the activities proposed. It should not come as a surprise that assumptions about mathematics curricula made in a time prior to the integration of technology in the classroom are, in some cases, no longer valid. Thus topics such as optimization, different matrix applications, linear and nonlinear regression, recursion etc. are now accessible to students in secondary and introductory college levels (prior to calculus).

Our goal in this article is to review some examples of applications, not yet present in most precalculus books that illustrate:

1. How the numerical and graphical capabilities of GC can be used to enhance the teaching and learning of key calculus concepts (functions, approximation, optimization…) at the secondary level.
2. How some of these concepts can be presented, using different representations, in the way they were developed and are better understood, i.e., via approximations.
3. Since technology enable students to revisit problems from different perspectives based upon the depth of their mathematical knowledge, it is possible to use a spiral approach to some of these concepts, like optimization, through different courses preceding calculus [Quesada & Edwards, 2005].

Let us consider for example the study of functions. The integration of technology allows:

1. The study of basic transformations, such as $f(x)+a, f(x+a), -f(x), a\cdot f(x), f(ax)$, hence we can consider families of functions, each with a root or parent function.
2. To introduce the analytical, graphical, and numerical aspects of functions simultaneously. Hence, students can support graphically and/or numerically analytical solutions and vice versa, whenever possible.
3. To study in addition of the traditional properties considered for every family of functions the following properties:
    a. determining irrational zeros, hence all the real zeros,
    b. finding local extrema, with intervals where the function is increasing or decreasing,
    c. using sequences to explore the local and end behavior,

d. comparing relative growth of functions from different families,
e. considering relevant examples of data that can be modeled via regression by the family of functions studied,
f. optimization problems.

Traditionally, students have been taught to complete the square of a quadratic expression. However, the emphasis generally was placed on justifying the quadratic formula or in other algebraic lessons derived from this process. Often, the graphical connection between the functions $y=(x-a)^2+b$ and $y=x^2$ was omitted. Similarly, one rarely finds in the basic books any reference to the transformations needed to obtain, for example, the rational function $f(x)=\frac{5x-11}{x-2}$ from g(x)=$\frac{1}{x}$, or $y=\ln 2(x-1)^3$ from $y=\ln x$. With the exception of trigonometric functions, very little attention was paid to the role of transformations on the basic families of continuous functions studied. As the next example illustrates graphing technologies facilitate the study of these transformations and their implications.

Example 1. What transformations are needed to obtain $f(x)=\frac{5x-11}{x-2}$ from $g(x)=\frac{1}{x}$. By simple division we can express $f(x)=5-\frac{1}{x-2}$, and follow the changes of the graph of *g(x)* via the horizontal shifting, the reflection, and the vertical shifting as seen in figures 1.a-1.c, and understand their effect on its properties such as the asymptotes of *f(x)*.

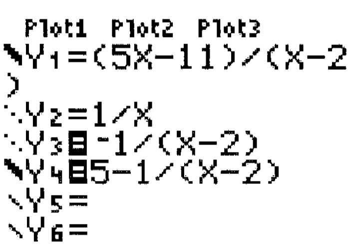


Figure 1.a

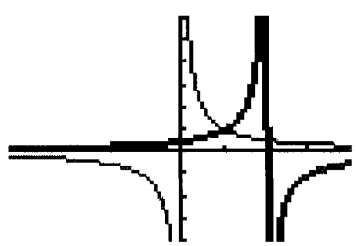


Figure 1.b

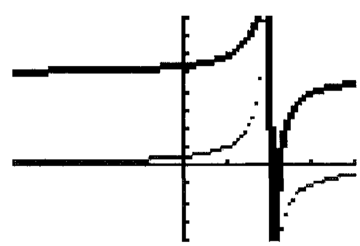


Figure 1.c

Our next example deals with the local behavior of a function. Purposely we have chosen an example which cannot be solved with the traditional techniques seen in calculus. Yet we will see that technology facilitates the study of this topic numerically. This method is far more intuitive, can be used for any type of functions, is independent of ready made formulas, and can be implemented using different approaches from elementary (table) to more sophisticated one such as recursion..

Example 2. How does the function $f(x)=\dfrac{|x+2|}{|1-\sqrt{x+3}|}$ behaves as $x \to -2$?

The numerical approach consists in generating two lists converging to -2 from both sides. For simplicity we use $\{-2 \pm 10^{-n}\}$ as $n \to \infty$. The simplest way is using a table (figure 2.a). The use of lists generated with sequences is depicted in figures 2.b and 2.c.

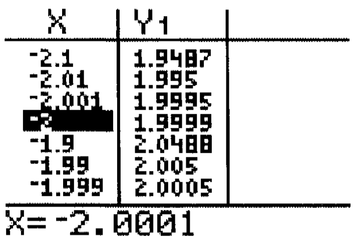


Figure 2.a

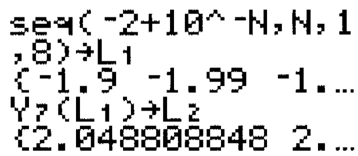


Figure 2.b

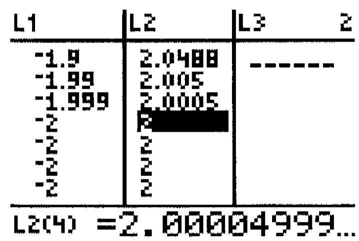


Figure 2.c

Finally, since we believe that recursion should be part of the secondary student toolbox [Quesada, 1999], in figures 2.d and 2.e we use lists generated recursively in the *Homescreen*.

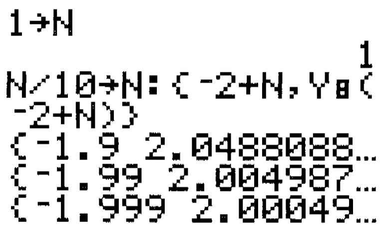


Figure 2.d

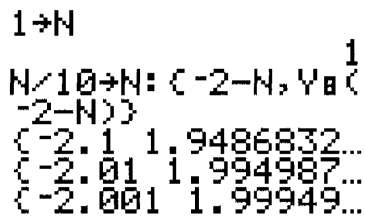


Figure 2.e

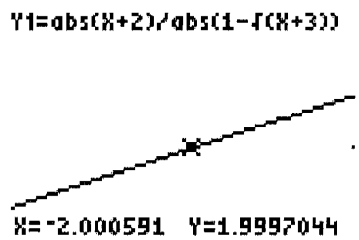


Figure 2.f

Asking students to confirm graphically their numerical results reinforces the interplay of both representations. Figures 2.f and 2.g confirm our previous result via zoom-in.

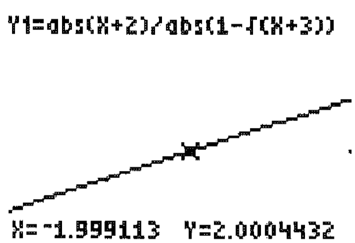


Figure 2.g

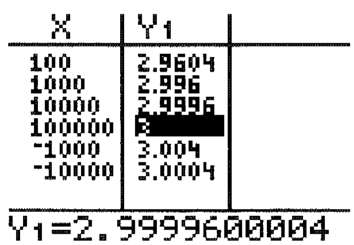


Figure 3.a

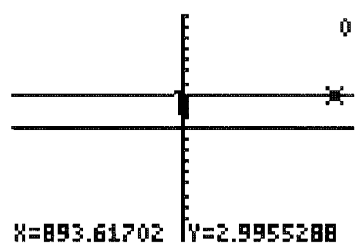


Figure 3.b

Example 3. How the function $y=\dfrac{3x-1}{x+1}$ does behave as $x \to \pm\infty$?

In the case of the global behavior we evaluate the function at the divergent sequences $\{\pm 10^n\}$ as seen in figure 3.a, or choose an appropriately large window (figure 3.b).
Example 4. As an immediate application of the global behavior figures 4.a and 4.b depicts both graphically and numerically that $\left(1+\frac{1}{x}\right)^x \to e$ as $x \to \infty$. We caution that some students in the pursue of graphical convergence may choose a window exceeding the precision of the calculator obtaining the chaotic behavior shown in figure 4.c due to the round up error.

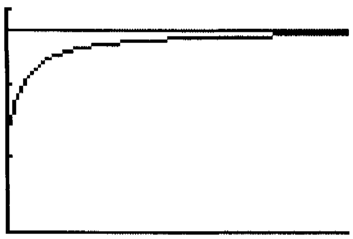
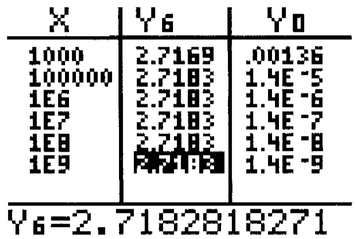


Figure 4.a



X	Y6	Y0
1000	2.7169	.00136
100000	2.7183	1.4E-5
1E6	2.7183	1.4E-6
1E7	2.7183	1.4E-7
1E8	2.7183	1.4E-8
1E9	2.7183	1.4E-9

Y6=2.7182818271

Figure 4.b

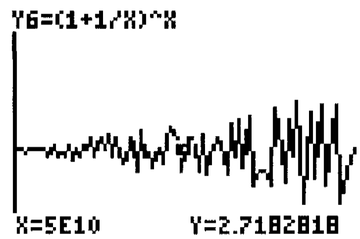


Figure 4.c

Example 5. Solve $x^4 = 2^x$.
Our last example illustrates the need for introducing the relative growth of different families of functions if transcendental equations are going to be studied. Understanding

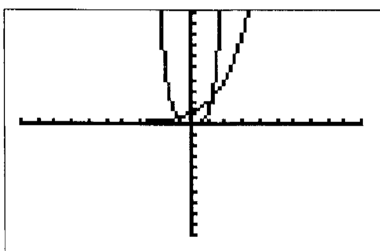


Figure 5.a

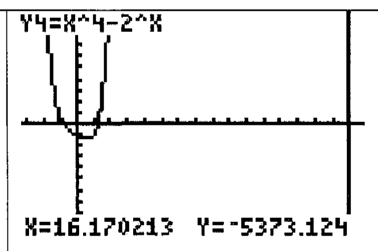
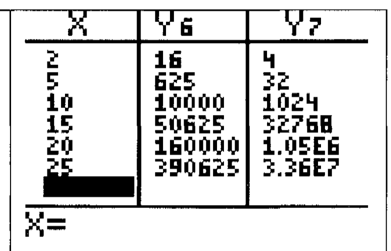


Figure 5.b



X	Y6	Y7
2	16	4
5	625	32
10	10000	1024
15	50625	32768
20	160000	1.05E6
25	390625	3.36E7

X=

Figure 5.c

that the exponential function grows faster than the power function is essential to guess the existence of a third root in this equation. However, as seen in figure 5.c, simply by using the table secondary students can easily decide which function grows faster.

References

1. Quesada, Antonio R. "Should Recursion be Part of the Secondary Student's Mathematics Toolbox?" The Intern. Journal of Computer Algebra in Math. Ed., Vol. 6, No. 2, pp. 103-116, UK, 1999.
2. Quesada, Antonio R., Edwards, Michael Todd . A Framework of Technology Rich Exploration. *The Mathematical Association of America, MathDL, Journal of Online Mathematics Applications (JOMA)*, June 2005.