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FINDING THE BRACHISTOCHRONE BETWEEN TWO POINTS IN A
VERTICAL PLANE BY DIRECT NUMERICAL INTEGRATION

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Abstract

In 1696, Bernoulli proposed determining the ‘curve of quickest descent’ between two points in a vertical plane. As is well known, the solution involves finding a function to make an integral an extreme. Here we verify this by numerically calculating the transit time between two points joined by various curves.

- I. Introduction
 - A. The brachistochrone problem
 - B. Properties of the cycloid
- II. Numerical integration of the time of descent of a particle along smooth curves in a vertical plane
 - A. The Cycloid
 - B. Straight Line
 - C. Parabola
 - D. Circle
- III. Suggested extensions for future work
 - A. The cycloid also a tautochrone
 - B. The dynamics of a sliding bead along a curve:
What happens if the curves have friction?
- IV. Conclusion

As is well known to math majors, Johann Bernoulli in 1696 proposed a challenge problem – the brachistochrone problem -- to his peers: Among all smooth curves joining two given points, to find that one along which a bead might slide, subject only to the force of gravity, *in the shortest time*.

The kinetic energy of the particle at the starting point $P_0(0,0)$ is zero, since it starts from rest. The work done by gravity in moving the particle from $(0,0)$ to any point (x,y) is mgy , the y -axis pointing downward, which equals the change in kinetic energy:

$$mgy = \frac{1}{2}mv^2 - \frac{1}{2}m(0)^2.$$

Thus the velocity $v = ds/dt$ that the particle has when it reaches $P(x,y)$ is $v = \sqrt{2gy}$.

Thus,

$$\frac{ds}{dt} = \sqrt{2gy}$$

$$dt = \frac{ds}{\sqrt{2gy}} = \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}{\sqrt{2gy}}.$$

And the time required for the bead to slide from P_0 to P_1 is given by:

$$T = \int_0^{x_1} \sqrt{\frac{1 + (f'(x))^2}{2gf(x)}} dx,$$

which depends on the curve $y = f(x)$ that passes through the points $P_0(0,0)$ and $P_1(x_1, y_1)$. The problem is to find that curve which *minimizes* the value of the integral above.

The solution involves finding a function that will make the integral of the time of descent an extreme value. The complete solution belongs properly to that branch of mathematics known as the calculus of variations. And the solution is the cycloid.

The parametric equations of the cycloid may be derived readily from the geometric properties of the curve. If a wheel of radius a is allowed to roll along a horizontal line without slipping, and we follow the curve traced by a point on the rim of the wheel, such a curve is the *cycloid*:

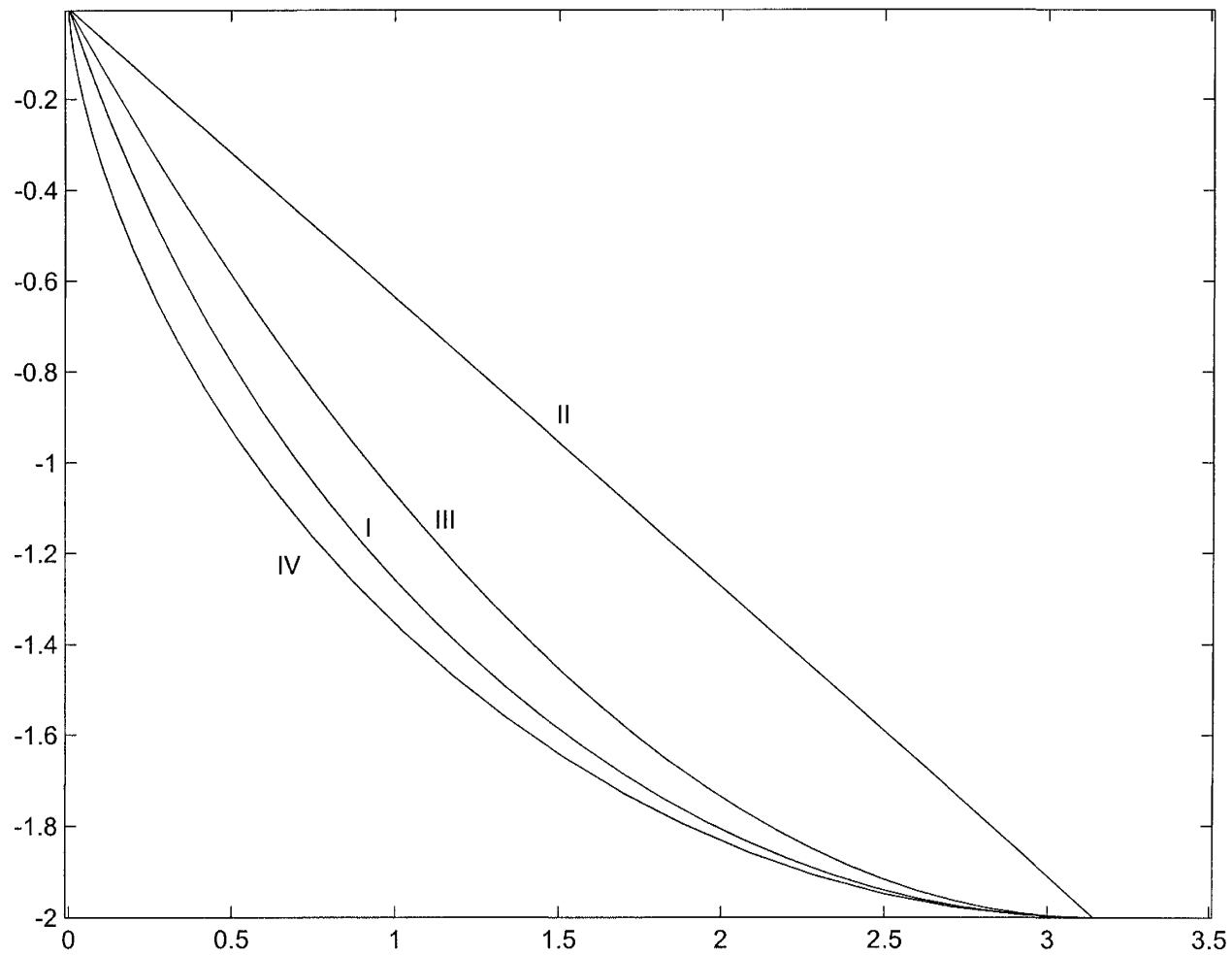
$$x = a(\phi - \sin \phi), \quad y = a(1 - \cos \phi)$$

where the parameter ϕ is the angle through which the initial vertical radius has rotated as the wheel rolls.

This paper will verify the solution by direct numerical calculation of the integral T . The initial point is chosen as the cusp and the final point is the lowest point of the arch of the cycloid. The parametric equations of the cycloid allow a closed form of the integral, $T_I = \pi\sqrt{a/g}$. The result is compared with three other familiar paths (see Figure): a straight line path, $T_{II} = T_I(1.185)$ a parabolic arc, $T_{III} = T_I(1.160)$, and a circular arc, $T_{IV} = T_I(1.149)$. The results of the computation all verify the solution. Numerical integration and the curves were done using Matlab.

There are also a few other problems that suggest themselves when we deal with the brachistochrone. (1) We also know that the time required to reach the bottom starting from the top is the same time, starting from rest, from *any* intermediate point of the arch. That is, the cycloid is also a tautochrone, as well as a brachistochrone. (A closed integration is shown, for example, by Thomas, 1972.) This is not true for any other curve. We would like to verify this by direct numerical integration.

VARIOUS PATHS BETWEEN TWO POINTS



(2) A further question that we can try to answer in the future is to investigate the dynamics of a particle sliding along a curve in a vertical plane. It might be interesting to know how the particle picks up its acceleration as it starts its descent and how this is determined by the shape of its path. How does, for example, a longer path provide a shorter time to reach its destination? There is a need to analyze its equations of motion as a function of the curve's parameters. In this regard, we can ask the realistic question: what changes will be introduced when we allow friction in the problem, that is, how is the time of descent affected if the curves are now rough? Will the cycloid still give the shortest time of descent? This problem may be of practical application if we install slides of various shapes in our recreational parks.

For the case of frictionless fall along curves joining two points in a vertical plane, we can safely conclude that the cycloid provides the quickest descent. When Bernoulli presented the brachistochrone problem to his colleagues he gave them six months to solve the problem, but after that time no solution was offered. Newton, who was then the Keeper of the Mint, learned of the problem later, and he solved the problem in a few hours and presented the solution anonymously. Bernoulli, upon seeing the solution, remarked: *I recognize the lion by his paw.*

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