Using JavaScript to Study the Stability of Time-Varying ARMA Models

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Using JavaScript, the stability of autoregressive moving-average models (ARMA) models are studied. ARMA models are used in areas such as: approximating differential equations, econometrics, studying the stability of state-space systems, and modeling the input-output relationships and designing control laws in systems engineering.

In undergraduate (and graduate) courses, when ARMA models are studied, the time-invariant case is usually considered-where the coefficients of the systems are constant functions of time. In this paper, a program is written to allow students to study the stability of such systems where there are perturbations in the coefficients-thus creating a time-varying system. The bounded-input bounded-output (BIBO) stability of the system will be studied by observing the output of the system and well as computing the time-varying zeros [1], [2] of the system.

Our goal will be to show that even small perturbations in the coefficients of AMRA models can lead to large changes in the outputs of the equations. Hence students need to be aware of the need for accuracy in deriving their models and the affects of noise in the systems.

We shall look at single-input single-output (siso) systems of the form:

$$y(k)+\sum_{j=1}^{n}\alpha_{n-j}(k)y(k-j)=u(k) \qquad (1.1)$$

To study the stability of systems of the form (1.1) we will use an operational algebra for time-varying systems which is described as follows: Where $a(k) \in IR\ \forall\ k \in Z$ define the operators $z^{-i}\bullet$ and $z^{-i}\circ$ [1], [2] by

$$z^{-i} \bullet a(k) = a(k-i) \tag{1.2}$$

and

$$z^{-i} \circ a(k) = a(k-i)z^{-i} \tag{1.3}$$

Using these operators (1.1) may be written in the form

$$(1+\sum_{j=1}^{n} \alpha_{n-j}(k) z^{-j}) \bullet y(k) = u(k) \tag{1.4}$$

Factor the time-varying polynomial defined by the left-hand side of (1.4) (such a factorization exists generically) into a product of 1st order linear factors as follows:

$$1+\sum_{j=1}^{n} \alpha_{n-j}(k) z^{-j} = (1-\delta_1(k)z^{-1}) \circ (1-\delta_2(k)z^{-1}) \circ \cdots \circ (1-\delta_n(k)z^{-1}) \tag{1.5}$$

Where $\delta_i(k) \in R\ \forall k \in Z\ i = 1...n$. The $\delta_i(k)$ are called the time-varying poles [2] of the n^{th} order polynomial (1.5).We now relate the BIBO stability of time-varying difference equations to their time-varying poles:

<u>Theorem 1.1 ([1], [2])</u>: A sufficient condition for a difference equation (1.1) to be BIBO stable is that its time-varying poles satisfy the condition $|\delta_i(k)| < \delta < 1$ i=1, 2,…, n $\forall k \in Z$.

Note 1: Theorem 1.1 is a generalization of the time-invariant result: a siso ARMA system is BIBO stable if all its roots lie inside the open unit disk.

Note 2: Unlike the time-invariant case, the stability of a time-varying system does not require all it's zeros to be bounded inside the unit disk at all times. As this goes beyond the scope of this paper, I refer the interested reader to [1] and [2].

Here we shall study 2^{nd} AMRA systems of the form

$$y(k)+[[\alpha_1]+[p_1]]y(k-1)+[[\alpha_2]+[p_2]]y(k-2)=[[\beta_1]+[p_3]]u(k) \tag{1.6}$$

where α_1, α_2, and β_1 are constants and p_1, p_2, and p_3 are perturbations which are defined by:

$$p_i = \text{Math.random()} * p_i - \frac{p_i}{2} \qquad (1.7)$$

where Math.random() is a randomly generated number between 0 and 1 for each time k.

ARMA systems of the form (1.6) have coefficients which are time-varying perturbations of time-invariant coefficients.

The program interface is shown in Figure 1: the outputs are the y(k) values for y(0) …y(number of iterates), and zeros1 and zeros2 are the time-varying zeros described above.

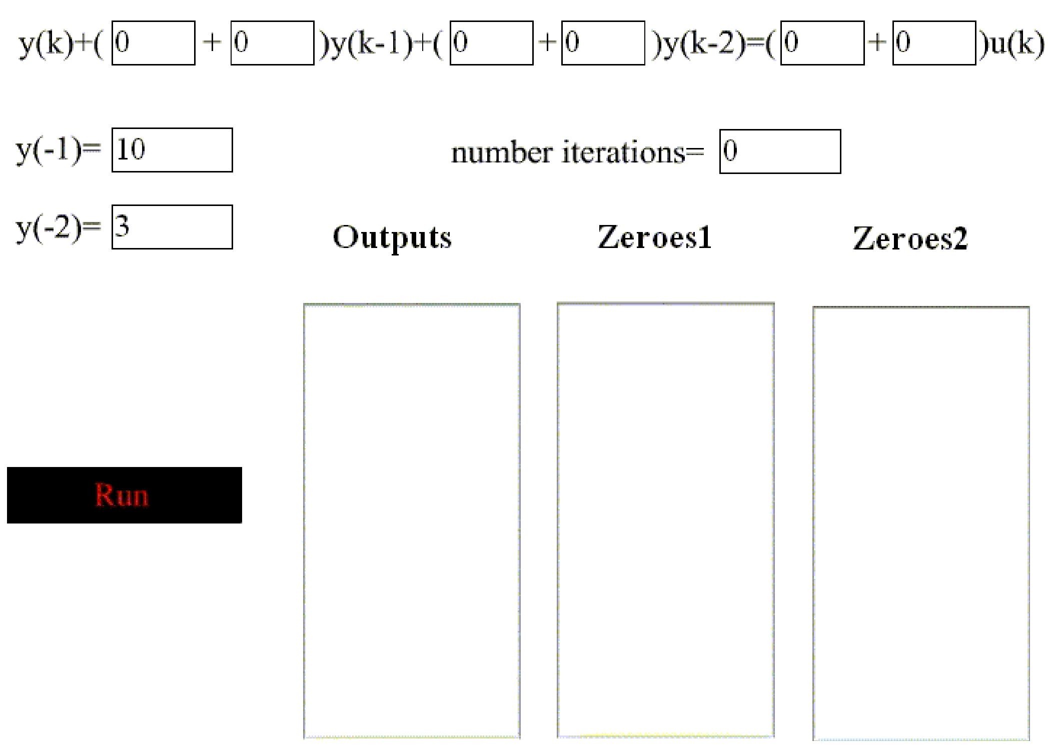

Figure 1-The Basic Program Interface

In this program initial values of the zeros were given as 1.

The first problem we look at is the system

$$y(k)-1.5y(k-1)+.56y(k-2)=0 \qquad (1.8)$$

with initial conditions y(-1)=10 and y(-2)=3. As is easily shown by geometric series, the output decays to 0. As shown in Figure 2 the time-varying zeros converge to the true zeros of .7 and .8.

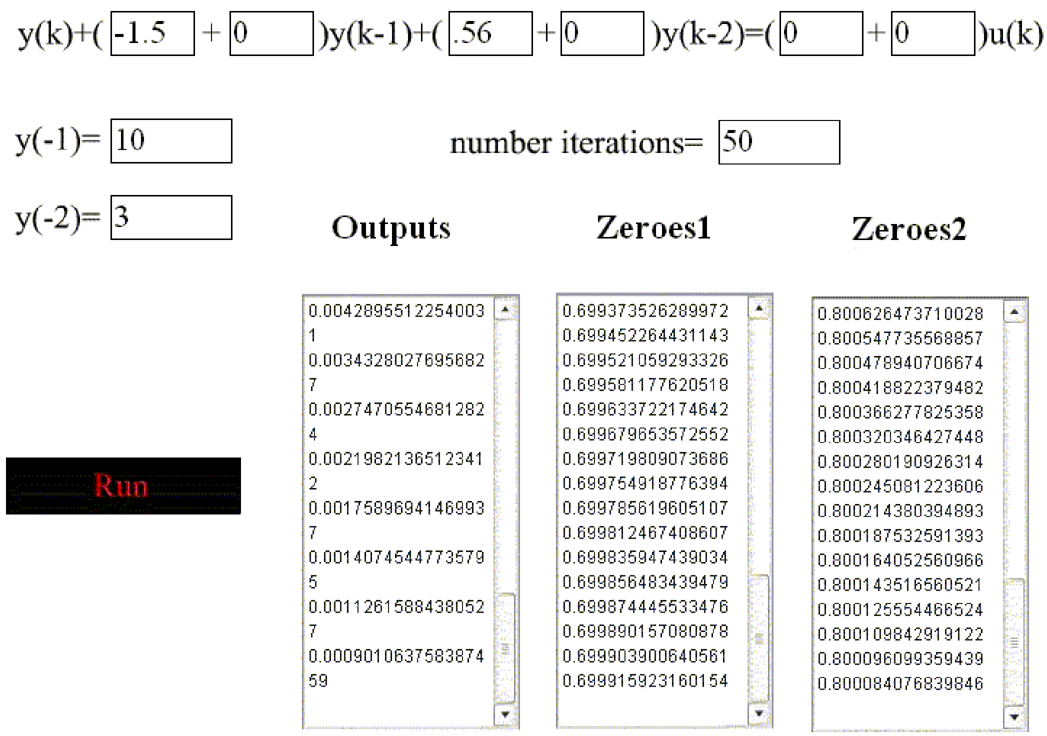

Figure 2-A Stable System with Time-Invariant Coefficients

We next look at the system

$$y(k)-1.5y(k-1)+.56y(k-2)=1 \qquad (1.9)$$

with the same initial conditions. The result, as shown in Figure 3, shows the output converging to $16\frac{2}{3}$.

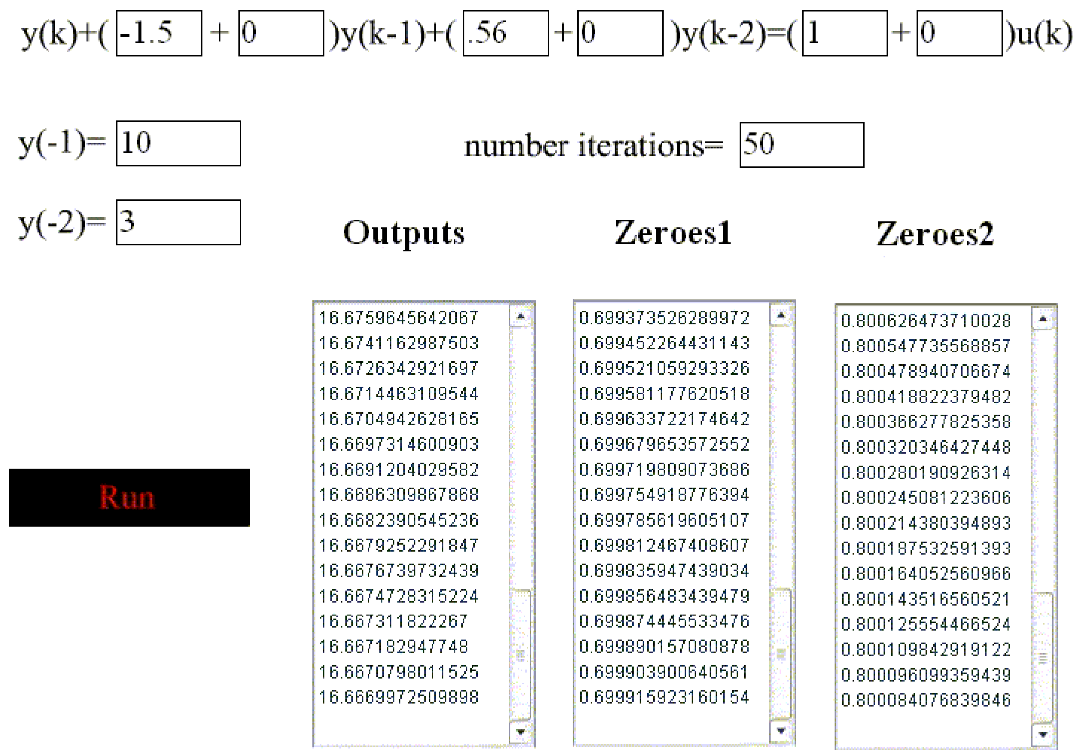

Figure 3-A Nonzero Input

Next we show how small changes in the coefficients can lead to large relative changes in the outputs.

$$y(k)-1.51y(k-1)+.56y(k-2)=1 \tag{1.10}$$

By changing the coefficient of y(k-1) from -1.5 to-1.51 the output now converges to 20, instead of $16\frac{2}{3}$ (Figure 4). A less than 1% change in the coefficient of y(k-1) leads to a 20% change in the output.

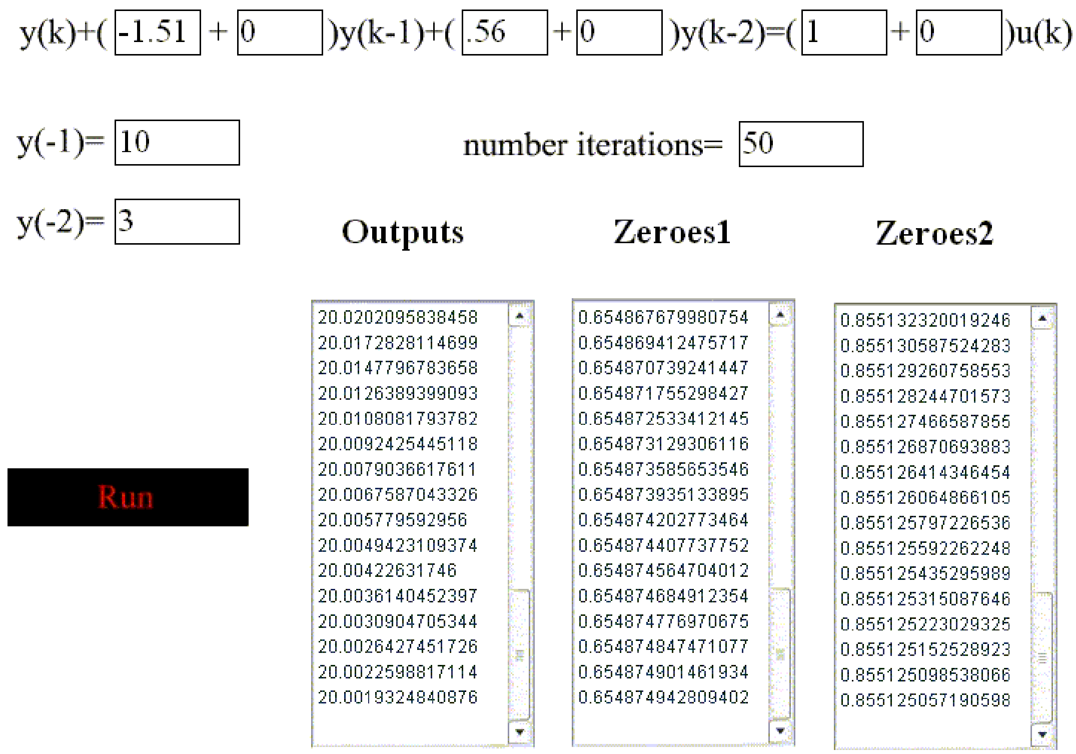

Figure 4-A Small Change in the Coefficient of y(k-1) Leads to Large Changes of Output

Finally we introduce a small amount of noise (±.05*Math.random()) on the coefficients of y(k-1) and y(k-2). Figure 5 shows how the output now diverges to infinity and that the zeros are outside the unit disk. Hence even small time-varying perturbations in the coefficients of a stable time-invariant system can cause it to become unstable.

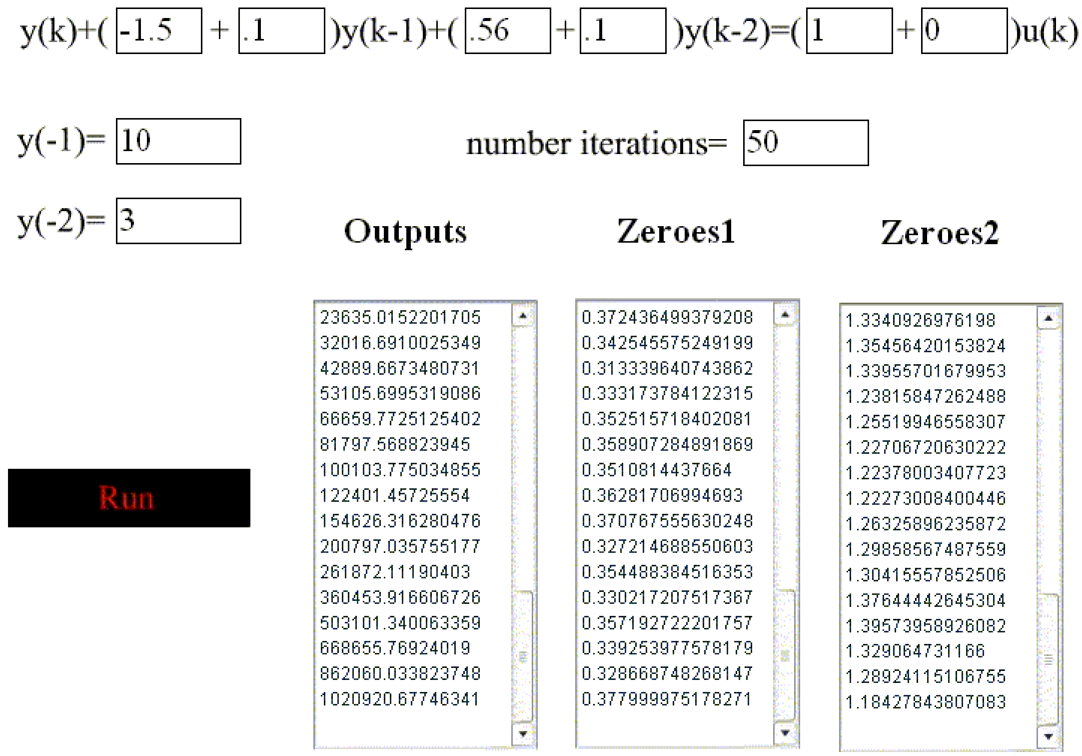

Figure 5-A Small Amount of Noise is Added to the System

Conclusion:

This program allows students to check that small disturbances in the coefficients of systems can alter the stability and convergence properties of the system. The study of the affects of such noise is generally covered in upper level undergraduate courses in numerical analysis and engineering courses.

References:

[1] "A Stability Criteria for Discrete-Time MIMO ARMA Models with Time-Varying Coefficients," *Proceedings of the 35th Annual Allerton Conference on Communication, Control, and Computing*, 1997, pp. 623-630, (with B. K. Ghosh).

[2] E. W. Kamen, "The Poles and Zeros of a Linear Time-Varying System," Linear Algebra and its Applications, 98: 1988, pp. 263-289.