OPTIMIZATION WITH TECHNOLOGY IN A MULTIVARIABLE CALCULUS COURSE (MINI-SYMPOSIUM: OPTIMIZATION IN MATHEMATICS)

William H. Richardson, Jr., Ph.D.
Francis Marion University
Department of Mathematics
PO Box 100547
Florence, SC 29501
wrichard@fmarion.edu

A demonstration of the use of technology in teaching optimization in a multivariable calculus course comprised one segment of the mini-symposium entitled *Optimization in Mathematics* at the *17th Annual International Conference on Technology in Collegiate Mathematics.* This paper provides a summary.

A one-semester course on linear programming is offered as an elective at Francis Marion University. Thus it is possible that a mathematics graduate's exposure to topics in optimization is limited to optimization in the single-variable setting during first-semester calculus and multivariable optimization later in multivariable calculus. When teaching the multivariable calculus course at Francis Marion University, little time is available for optimization due to the multitude of topics usually presented in one semester. An elective course in nonlinear programming is to be offered beginning in the next semester to address this problem.

In the mathematics department at Francis Marion University, MAPLE is used extensively to aid in teaching and problem solving. MAPLE (or any of several other powerful computer algebra systems) is particularly useful in teaching multivariable calculus because of its 3-dimensional graphics capabilities. In addition to plotting, MAPLE offers many sophisticated tools for numerical and symbolic determination or manipulation of various mathematical constructs in multiple dimensions such as vectors, dot products, cross products, tangent vectors, gradients, directional derivatives, partial derivatives, and multiple integrals, to name a few. Examples of a few of these follow.

Throughout the discussion of optimization in more than one variable, it is helpful to consistently refer students to the simpler corresponding concepts from first semester calculus with which they are already familiar. For example consider an introduction to terminology relating to optimization of functions of two variables. Critical numbers, relative and endpoint extreme values, and inflection points can be compared to stationary points, relative extreme values, boundary point extreme values, and saddle points. Three-dimensional (3D) plots are essential in helping students understand these concepts. Figure 1 shows a plot of $f(x,y)=\left(x^2+3y^2\right)e^{-\left(x^2+y^2\right)}$, which can be rotated to allow viewing from any perspective. This function presents a relative and absolute minimum at

(0,0,0), relative and absolute maxima at the points (0,-1,1.104) and (0,1,1.104), and saddle points at (-1,0,0.368) and (1,0,0.368). Figure 2 shows the trace of function $f(x,y)$ in the *x*-*z* plane and Figure 3 shows the trace in the *y*-*z* plane. In this example, MAPLE plots in the plane and in 3-space are used to ensure a thorough understanding of beginning concepts in multivariable optimization.

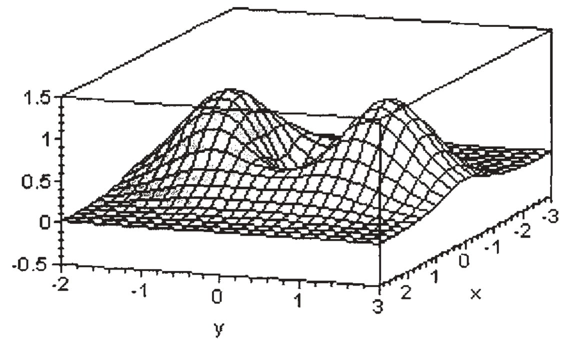

Figure 1. A plot of the function $f(x,y)=\left(x^2+3y^2\right)e^{-\left(x^2+y^2\right)}$

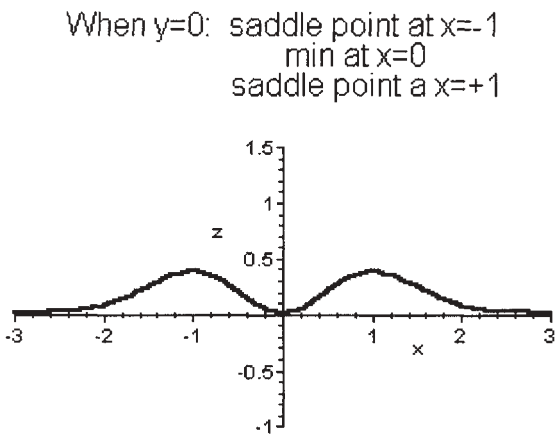

Figure 2. A plot of the trace of the function $f(x,y)=\left(x^2+3y^2\right)e^{-\left(x^2+y^2\right)}$ in the *x*-*z* plane.

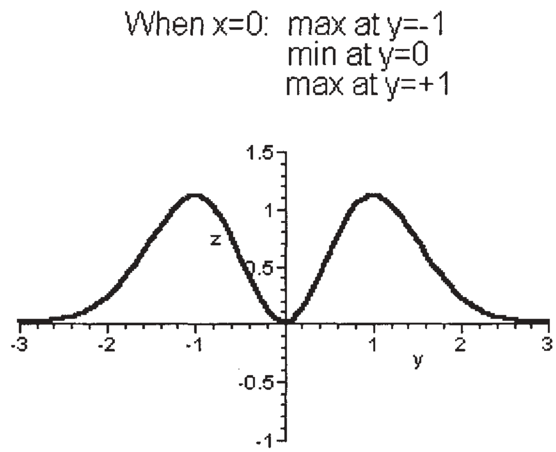

Figure 3. A plot of the trace of the function $f(x,y)=\left(x^2+3y^2\right)e^{-\left(x^2+y^2\right)}$ in the *y*-*z* plane.

Although students are required to demonstrate a thorough understanding of the second partials test and its use by solving problems using the pencil-and-paper method, use of MAPLE is also required. An example follows. In the steps shown in Figure 4, MAPLE is used to define the function $f(x,y)=x^3+y^2+2xy-4x-3y+5$. Subsequently $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y^2}$ are determined and rendered as functions of *x* and *y* in MAPLE. Finally, MAPLE's command *fsolve* is used to determine two simultaneous solutions to $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$. The result is two stationary points: $\left(1,\frac{1}{2}\right)$ and $\left(-\frac{1}{3},\frac{11}{6}\right)$.

```
> f:=(x,y)->(x^3+y^2+2*x*y-4*x-3*y+5):
> fx:=unapply(diff(f(x,y),x),x,y):
> fy:=unapply(diff(f(x,y),y),x,y):
> fxx:=unapply(diff(f(x,y),x,x),x,y):
> fxy:=unapply(diff(f(x,y),x,y),x,y):
> fyy:=unapply(diff(f(x,y),y,y),x,y):
> fsolve({fx(x,y),fy(x,y)},{x,y},{x=-3..3,y=-3..3});
> fsolve({fx(x,y),fy(x,y)},{x,y},{x=-3..0,y=-3..3});
```

$$\{x = 1.000000000, y = 0.5000000000\}$$

$$\{y = 1.833333333, x = -0.3333333333\}$$

Figure 4. MAPLE commands needed to find the stationary points for function $f(x,y) = x^3 + y^2 + 2xy - 4x - 3y + 5$

Next attention is turned to the determination of the nature of the stationary points. (See Figure 5.) MAPLE's command *hessian* is used to evaluate the Hessian matrix for $f(x,y)$ at the first of the two stationary points $\left(1, \frac{1}{2}\right)$. Subsequently, MAPLE's command *det* computes the determinant of the Hessian matrix evaluated at the stationary point. This value $f_{xx}f_{yy} - (f_{xy})^2$, which is called "bigD" below, is the well-known value of the

```
> hessian(f(x,y), [x,y]);
```

$$\begin{bmatrix} 6x & 2 \\ 2 & 2 \end{bmatrix}$$

```
> H:=subs({x=1.0,y=0.5},hessian(f(x,y), [x,y]));
```

$$H := \begin{bmatrix} 6.0 & 2 \\ 2 & 2 \end{bmatrix}$$

```
> bigD:=det(H);fxx(1,0.5);
```

$$bigD := 8.0$$

$$6$$

Figure 5. MAPLE commands needed to determine if a stationary point for function $f(x,y) = x^3 + y^2 + 2xy - 4x - 3y + 5$ corresponds to an extreme value or a saddle point.

second partials test. Since $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} > 0$, the first stationary point $\left(1, \frac{1}{2}\right)$ yields a relative minimum value for the function – namely, $\frac{7}{4}$. The second stationary point $\left(-\frac{1}{3}, \frac{11}{6}\right)$ was treated similarly, and a saddle point was found at $\left(-\frac{1}{3}, \frac{11}{6}, \frac{317}{108}\right)$. A plot of the function with arrows pointing out the relative minimum at $\left(1, \frac{1}{2}, \frac{7}{4}\right)$ and saddle point at $\left(-\frac{1}{3}, \frac{11}{6}, \frac{317}{108}\right)$ is shown in Figure 6.

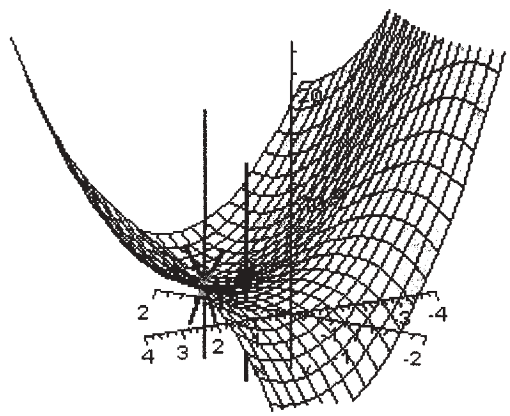

Figure 6. A plot of the function $f(x,y) = x^3 + y^2 + 2xy - 4x - 3y + 5$ showing locations for the relative minimum $\left(1, \frac{1}{2}, \frac{7}{4}\right)$ and saddle point $\left(-\frac{1}{3}, \frac{11}{6}, \frac{317}{108}\right)$.

In the next example, a MAPLE 3D plot is used to help students visualize and better understand an optimization problem in which boundary points must be considered. The problem is to find extreme values for the function $f(x,y) = x^2 - 2xy + 2y$ on or within the rectangular boundary described by: $\{(x,y) \mid 0 \le x \le 3, 0 \le y \le 2\}$. After solving this problem with pencil and paper, various MAPLE plots were generated, the last of which is shown in Figure 7.

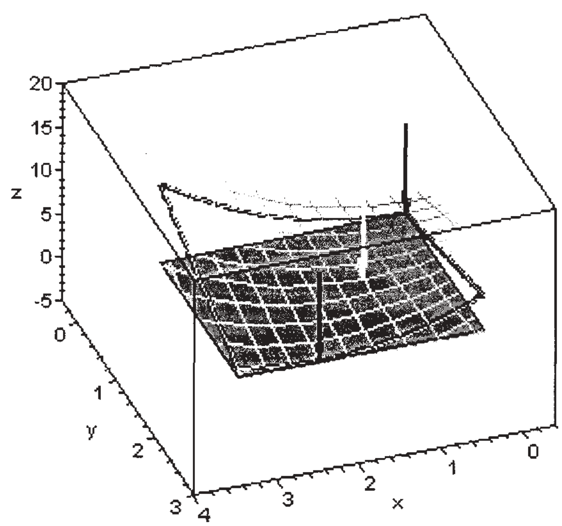

Figure 7. A plot of the function $f(x,y) = x^2 - 2xy + 2y$ showing the rectangular region in the *x-y* plane and pointing out four extreme values – three corresponding to boundary points and one to an interior point of the region.

Recall that, while using MAPLE, the plot can be rendered from any perspective. This helps students visualize the surface, the rectangular boundary lying below the surface, and even the projection of the planar, rectangular region onto the surface as shown in Figure 7.

As a final example, consider how MAPLE can be used to aid students in understanding the concepts underlying the use of Lagrange multipliers for optimization. The optimization problem is to find all extreme values for the function $f(x,y)=xy$ subject to the constraint $x^2+y^2=1$ using the method of Lagrange multipliers. Figure 8 shows a plot of the function, the constraint, and the corresponding space curve lying on the surface representing the function. Note that the user can make it visually easy to distinguish among the different plots by setting the colors of the components of the MAPLE plots. The plot in Figure 8 helps students clearly understand the objective of the optimization problem. Plots of the constraint in the *x-y* plane superimposed on level curves of the function are easy to create as well.

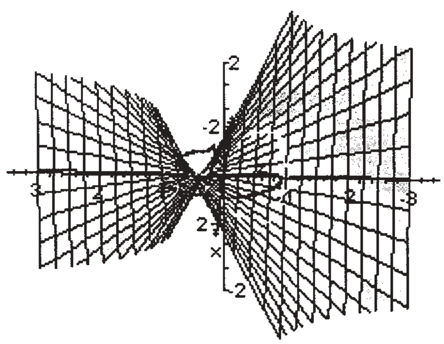

Figure 8. A plot of the function $f(x,y)=xy$, the constraint $x^2+y^2=1$ in the *x-y* plane, and the corresponding space curve on the surface.

Animated plots can also be generated using MAPLE. Such a plot for $f(x,y)=xy$ is shown in Figure 9. Although it cannot be shown here, the level curves shown in that figure are animated to appear one-at-a-time sequentially for decreasing values of $c \in \Re$ in $f(x,y)=c$. This helps students see that an extreme value for *f* is possible whenever a level curve first becomes tangent to the constraint curve – that is, when the gradient vector for the function and the gradient vector for the constraint are parallel. In this example, the function *f* assumes a maximum of 0.5 when $(x,y)=(.707,.707)$ and $(x,y)=(-.707,-.707)$. A similar animation was used to reveal a minimum of –0.5 when $(x,y)=(-.707,.707)$ and $(x,y)=(.707,-.707)$.

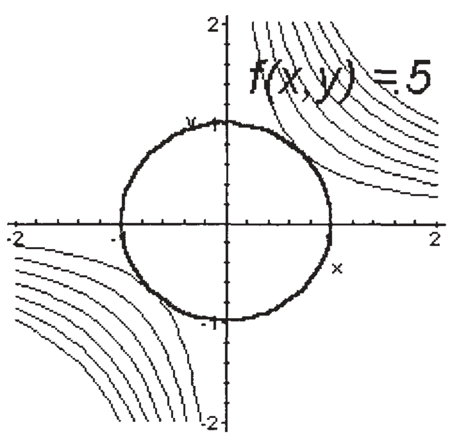

Figure 9. One of a sequence of plots comprising a MAPLE animation that shows level curves for $f(x,y)=xy$ first touching the constraint curve given by $x^2+y^2=1$.

These examples show how a computer algebra system like MAPLE can be used to greatly enhance visual presentations for the classroom when discussing optimization. The author has used similar graphics throughout a course in multivariable calculus and in other courses as well. The powerful mathematics tools offered by most computer algebra systems can also be used to solve problems that prove too difficult for pencil-and-paper solutions.