# USING GRAPHING CALCULATORS TO ENHANCE STUDENTS' UNDERSTANDING OF THE FORMAL DEFINITION OF LIMITS

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**Abstract**. The purpose of this study was to determine if the use of a graphical approach via the graphing calculator enhances student understanding of the formal definition of limit. College students in six sections of Calculus I participated by completing a quiz prior to the introduction of the definition, and a second similar quiz after covering this topic. Students in four sections received traditional instruction on the formal definition of limit, but in two sections an approach utilizing graphing calculators was incorporated into the instruction. Students who received graphical instruction performed significantly better on two of the conceptual items on the post-test, as compared to the students who received traditional instruction.

## Introduction

A review of existing research found that Cornu (1991), Gray and Tall (1994), Dubinsky (1991), Dubinsky & Tall (1991), Li and Tall (1992), and Kidron and Zehavi (2002) studied teaching the limit concept. Some of these researchers used computers to provide experiences to help students develop the cognitive foundations for the formal definition of a limit. In addition to the use of computers in improving the understanding of limits, the body of research on the effectiveness of graphing calculators in mathematics education has increased considerably in the last fifteen years [Burrill, et al, 2002] and [Ellington, 2003]. None of the studies reviewed dealt with the impact of graphing calculators on student understanding of the formal definition of limit. The purpose of our study was to determine if the introduction of the graphical approach via the graphing calculator enhances the students' understanding of the formal definition of limit.

The concept of the limit of a function at a given value is central to the study of calculus. The formal definition of the limit of a function, usually referred to as the $\varepsilon - \delta$ definition, is a difficult one for many students to grasp. Following the introduction of this concept, the typical problem consists in asking the student to find for any given value of epsilon the corresponding value of delta. Traditionally, the students follow a well-known algebraic approach consisting of solving absolute value inequalities to solve this problem. The algebraic difficulties inherent to this method not only restrict the examples considered to primarily linear functions and some quadratic functions, but also (in our opinion) focus the attention of many students on the algebraic manipulations needed rather than on the meaning of the result. From observations of our students' work we noticed the following mechanical and conceptual difficulties:

1. The use of quantifiers in the definition and their role in proving that the limit exists is new for the majority of the students.

2. Their previous work does not seem to prepare many of our students to clearly understand the interplay between the algebraic and graphical expressions of continuous inequalities. As a result, they struggle when trying to set up the necessary inequalities from the given limit, as well as when interpreting the meaning of the results obtained.
3. Many students lack skills when removing the absolute values and manipulating algebraically the inequalities to relate $\varepsilon$ and $\delta$. The degree of difficulty increases when non linear functions are considered.
4. Finally, even after the introduction of graphing technology during the last decade, many students have difficulty relating the algebraic and graphical representations of functions.

After years of experience using graphing calculators to introduce this concept we decided to conduct a study on the effectiveness of this approach.

**Procedure**

The students were given the pre-test in Figure 1 to assess their understanding of limits prior to the topic's introduction in the course. After the unit on limits had been covered, the students were given a comparable post-test. A total of 166 students completed the pre-test, and 159 took the post-test, but only 148 students completed both the pre-test and post-test. The six sections were selected based on the instructors' willingness to participate in the study. Two of the instructors agreed to allow one of us (with considerable experience on integrating technology) to introduce the formal definition of limit and related $\varepsilon-\delta$ problems using the graphical approach, working with a graphing calculator.

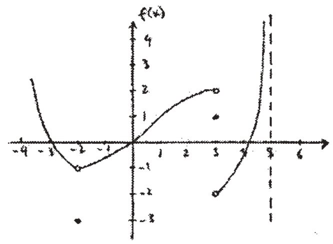
This researcher replaced the assigned classroom instructor for two class periods in one section, but only for one class period (50 minutes) in another section, depending on the willingness of the assigned instructors. These two sections were considered "Experimental" sections, whereas the other participating four sections will be referred to as "Control" sections. Students in the control sections were receiving traditional instruction on limits, without the use of graphing technology. A comparable amount of time was spent on reviewing absolute value inequalities and in presenting the formal definition of limit in all six sections. To help this situation, in all six sections, students received a one-page handout reviewing these concepts. In the experimental section in which the researcher intervened for two class periods, part of the first period was devoted to the teaching of such concepts. In the other experimental section, there was little time available to touch on these topics. In both experimental sections, students were provided with a handout detailing the graphical approach together with the steps needed to solve a typical problem with the aid of a graphing calculator and an example (Fig. 2).

**Results**

The pre- and post-test each consisted of five similar questions (Figure 1). On the pre-test, only a small percentage of the students were able to provide correct answers to the limit definition questions (Questions 2-5). For example, only 3 (1.8%) of the 166 students who completed the pre-test got the correct answer to Question 2. By comparison, 21 (12.7%), 30 (18.1%), and 19 (11.5%) gave correct answers to Questions 3, 4, and 5, respectively, on the pre-test.

Calc I Section#:_____ Pre-test Fall 2003 Name:____________________

1. Use the graph below for the following questions:




a. $\lim_{x\to -2} f(x) =$ _____

b. $\lim_{x\to 3^+} f(x) =$ _____

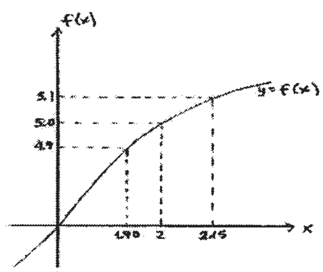
c. $\lim_{x\to 3} f(x) =$ _____

d. $\lim_{x\to 5^-} f(x) =$ _____

e. $f(-2) =$ _____

2. Given the graph below, what is the largest value of $\delta$ so that $|f(x)-5| < 0.1$ whenever $|x-2| < \delta$ ?

$\delta =$ ______




For questions 3-4 circle the best possible answer.

3. In using the $\varepsilon, \delta$ definition to prove that $\lim_{x\to 2}(2x-1) = 3$, when $\varepsilon$ is *0.5*, what is the largest value that $\delta$ can have?

A) *0.5* B) *0.25* C) *1* D) *0.2* E) *2*

4. Let $\lim_{x\to 3}(4x+10) = 22$. If $\varepsilon = 0.2$, what is the largest value that $\delta$ can have?

The statement above can be restated as:

I) How close to *3* do we need to take *x* so that ( $4x+10$ ) is within a distance of *0.2* from *22*.

II) How close to *22* do we need to take *x* so that ( $4x+10$ ) is within a distance of *0.2* from *3*.

III) Find the largest $\delta$ such that $|(4x+10)-22| < 0.2$ whenever $|x-3| < \delta$.

A) I only B) II only C) III only D) I and II E) I and III

5. In using the $\varepsilon, \delta$ definition to prove that $\lim_{x\to 1} x^2 = 1$, when $\varepsilon$ is *0.25*, what is the largest value can have? (Choose the closest possible answer)

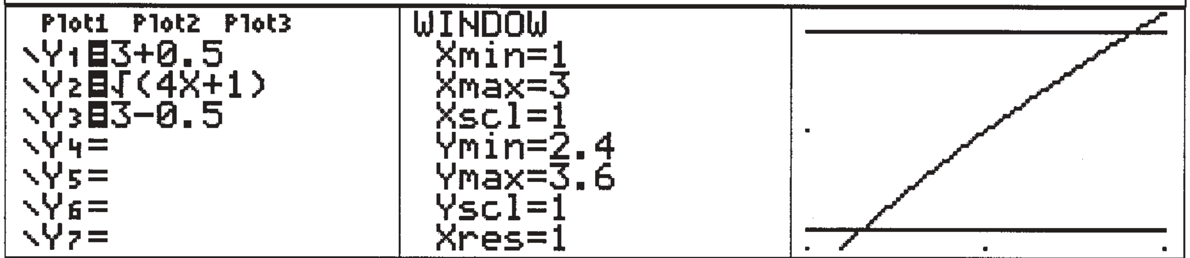
A) *0.134* B) *0.25* C) *1* D) *0.118* E) *0.2*

Figure 1

On the post-test, the percentage of correct responses increased considerably on all of these questions, as anticipated. Of the 159 students who completed the post-test, the number of correct responses rose to 112 (70.4%) for Question 2, 77 (48.4%) for Question 3, 111 (69.8%) for Question 4, and 101 (63.5%) for Question 5. Question 1 consisted of five parts. The average number of parts done correctly was 4.1 on the pre-test (n=166) and 4.4 for the post-test (n=159).

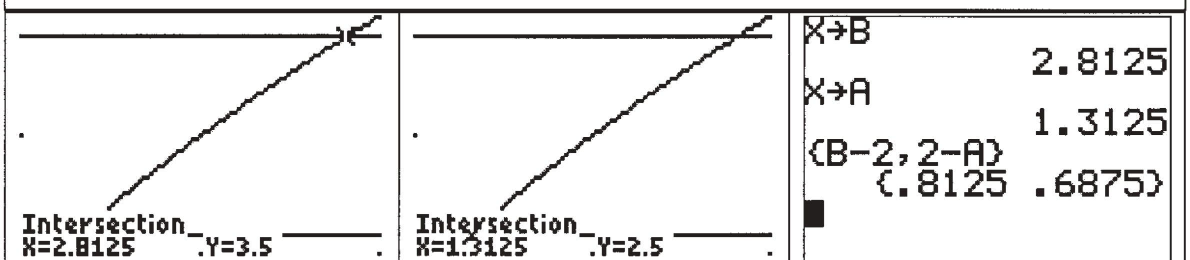
*Example*. $\lim_{x \to 2} \sqrt{4x+1} = 3$. Find $d$ for $e = 0.5$.

*Solution*. Screens 1-3 show the three functions (step 1), the window (step 2), and the graph obtained.




Screen 1 | Screen 2 | Screen 3

Next we use 2nd CALC to find the interceptions of $y_2 \cap y_1$ and $y_2 \cap y_3$ (step 3) as seen in figures 4-5. Storing the $x$-coordinates of the interception points immediately after they are calculated precludes using the pencil while allowing using all the decimals obtained internally by the calculator. The solution (Screen 6) is $d = \min(B-2, 2-A) = 0.6875$.




Screen 4 | Screen 5 | Screen 6

Try now on your own to find $d$ first for $e = 0.1$, next for $e = 0.05$.

Figure 2

A preliminary analysis of the data showed that the results for the four sections designated as "Control" were similar on all questions, as were the results for the two "Experimental" sections. Therefore, for subsequent analysis, the data for all four control sections were combined, as well as for the two experimental sections. The chi-square test was used to compare the students from the control sections to those in the experimental sections on their answers to Questions 2-5. On the pre-test, as Table 1 illustrates, the control students did significantly better on Question 3, and slightly better (but not significantly so) on Question 5. The experimental and control students didn't differ much at all in terms of their success on Items 2 and 4 of the pre-test. On the post-test, as Table 2 shows, students in the experimental sections did significantly better on Questions 2 and 3 than students in the control sections.

| | Percent giving correct response on Pre-test | | | |
|---|---|---|---|---|
| | Question 2 | Question 3 | Question 4 | Question 5 |
| Control (n=91) | 2.2 | 17.6 | 17.6 | 15.4 |
| Experimental (n=75) | 1.3 | 6.7 | 18.7 | 6.7 |
| p-value for chi-sq. test | .677 | .030 | .857 | .072 |

Table 1

| | Percent giving correct response on Post-test | | | |
|---|---|---|---|---|
| | Question 2 | Question 3 | Question 4 | Question 5 |
| Control (n=88) | 62.5 | 36.4 | 68.2 | 65.9 |
| Experimental (n=71) | 80.3 | 63.4 | 71.8 | 60.6 |
| p-value for chi-sq. test | .015 | .001 | .618 | .486 |

Table 2

On Questions 4 and 5, the differences between the experimental and control results were not statistically significant. A comparable analysis, using only the data from the 148 students who completed both the pre-test and post-test, produced virtually the same results as shown below.

**Discussion of Findings**

It is the researchers' belief that a graphical approach, as enhanced by the use of a graphics calculator, can aid the student in grasping the epsilon-delta definition of limit and solving standard $\varepsilon-\delta$ problems. The results of this study tend to support the researchers' belief. In particular, Questions 2, 3 and 5 directly assess the student's understanding of the epsilon-delta definition and solving standard $\varepsilon-\delta$ problems. (Question 1 addresses previous knowledge on limits, and Question 4 deals with translating the definition using algebraic expressions.) It was on Questions 2 and 3 that students in the experimental sections outperformed those receiving the traditional instruction on limits. The researchers also anticipated that the experimental approach would lead to better results on Question 5, for which a statistically significant difference between the two approaches was not found. However, the lack of a difference on this item may be due to the specific question posed on the post-test. After the quiz had been given, the researchers realized that one of the choices (Choice D – answer =1), would be a likely guessed answer, since the value 1 appears in the question in two places. Moreover, Question 5 deals with a nonlinear function, yet more students in the control sections got the correct answer to that question (65.9%) than for the seemingly easier linear function in Question 3 (36.4%).

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