Optimization in Mathematics

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This talk and subsequent paper covers using mathematical modeling and power of MAPLE to solve these modeling problems. During the actual talk, MAPLE programs were illustrated to solve the various modeling scenarios discussed in this paper. The programs and code are too long to be a part of this paper. Many of our optimization programs have been published on the MAPLE applications web site but I am more the willing to send you our work. The following programs or templates have been developed for optimization purposes for single and multivariable optimization for both constrained and unconstrained optimization:

- Modeling with Simple Least Squares
- Modeling with Non-Linear Least Squares
- Golden Section Search Method
- Fibonacci Search Method
- Newton's Method for Critical Points
- Steepest Ascent Method
- Newton Raphson Multivariable Method (2 variable method)
- Conjugate Direction Method (Fletcher, Davidson, Reeves)
- LaGrange Multipliers with 2-D, 3-D Visuals
- Kuhn-Tucker with 2-D,3-D Visuals
- Simplex Method with Tableaus

Again, only a few of these can be illustrated in this paper.

PROJECT 1:

Using the model fitting criterion, minimize the sum of the absolute deviations, fit the following model, $y=cx^2$, to the data

x	1	2	3
y	2	5	8

The model form is Minimize $\Sigma\ |y_i-cx_i^2|$ (1)
and eq (1) can be written as
Minimize $|\ 2-c| + |\ 4 - 4\ c| + |8 - 9\ c|$

Now, we know the calculus cannot be used to solve this problem. We turn to Golden Section search. Golden Section Search is a search procedure that utilizes the *golden ratio, 0.618 and 0.382.*

In order to use the Golden Section search procedure, we must insure that certain assumptions hold. These key assumptions include:

(1) the function must be unimodal over a specified interval,
(2) the function must have an optimal solution over a known interval of uncertainty, and
(3) we must accept an interval solution since the exact optimal cannot be found by this method.

Only an interval solution, known as the final interval of uncertainty, can be found using this technique. The length of this final interval is controllable by the user and can be made arbitrarily small by the selection of a *tolerance value*. The final interval is guaranteed to be less than this tolerance level.

Finding the maximum of a function over an interval

This search procedure to find a maximum is iterative, requiring evaluations of $f(x)$ at experimental points x_1 and x_2, where $x_1 = b - r(b - a)$ and $x_2 = a + r(b - a)$. These experimental points will lie between the original interval $[a, b]$. These experimental points are used to help determine the new interval of search. If $f(x_1) < f(x_2)$ then the new interval is $[x_1,b]$ and if $f(x_1) > f(x_2)$ then the new interval is $[a,x_2]$. The iterations continue in this manner until the final interval length is less than our imposed tolerance. Our final interval contains the optimum solution. It is the size of this final interval that determines our accuracy in finding the approximate optimum solution. The number of iterations required to achieve this accepted interval length can be found as the smallest integer greater than k where k equals [1,4]:

$$k = \frac{\ln(\frac{tolerance}{(b-a)}}{\ln(0.618)}.$$

Often we are required to provide a point solution instead of the interval solution. When this occurs the method of selecting a points is to evaluate the function, $f(x)$, at the end points of the final interval and at the midpoint of this final interval. For maximization problems, we select the value of x that yields the largest $f(x)$ solution. For minimization problems, we select the value of x that yields the smallest $f(x)$ solution.

The algorithm that was used to produce our MAPLE code is shown in Figure 2.

To find a maximum solution to given a function, $f(x)$, on the interval $[a, b]$ where the function, $f(x)$, is unimodal.

INPUT: endpoints a, b; tolerance, t
OUTPUT: final interval $[a_i, b_i]$, $f(midpoint)$
Step 1. Initialize the tolerance, $t > 0$.
Step 2. Set r=0.618 and define the test points

$$x_1 = a + (1-r)(b-a)$$
$$x_2 = a + r(b-a)$$

Step 3. Calculate $f(x_1)$ and $f(x_2)$
Step 4. Compare $f(x_1)$ and $f(x_2)$

a. If $f(x_1) \leq f(x_2)$, then the new interval is $[x_1, b]$:
a becomes the previous x_1
b does not change
x_1 becomes the previous x_2
Find the new x_2 using the formula in Step 2.

b. If $f(x_1) > f(x_2)$, then the new interval is $[a, x_2]$:

a does not change

b becomes the previous x_2

x_2 becomes the previous x_1

Find the new x_1 using the formula in Step 2.

Step 5. If the length of the new interval from Step 4 is less than the tolerance specified, the stop. Otherwise go back to Step 3.

Step 6. Estimate x^* as the midpoint of the final interval and compute, $f(x^*)$, the estimated maximum of the function.

STOP

Figure 2. Golden Section Algorithm

PROJECT 2. Planning and Production Control

Introduction.

Many optimization problems require the simultaneous consideration of a number of independent variables. In planning and producing items, one must consider many factors that impact in the process. The company might desire to maximize profit, minimize cost, maximize production levels, improve efficiency, minimize shipping time, and a host of other options. Many of these can be solved by the following techniques:

Differential Calculus
Lagrange Multipliers
Linear Programming
Dynamic Programming

We will cover most of these methods during this course except linear programming where we devote an entire semester course to the subject.

During this block we studied only two numerical search techniques for multivariable functions:
Gradient Search(Ascent and Descent)
Newton-Raphson Method

Key definitions and variables:
z is the measure of performance.
$x_1, x_2, \ldots, x_n$ are the inputs that effect z.
Optimum point: The values of $x_1, x_2, \ldots, x_n$ which maximize or minimize z.
Optimal Value. The value of z for the optimum point.
Unimodal: Most search strategies rely on the assumption that the surface is unimodal, that is, it only has one peak over the region of concern.

Part I.
You are hired as a consultant and ask to optimize all facets of their planning and production.

RABA manufactures 15" color TV sets. The company plans to improve their building of color Tvs. It appears as though a new chip added to the circuitry will improve reception and survivability of the TV. The new chip is extremely sensitive and must be continuously monitored. The monitoring process is assumed to be modeled by the expression:

$Y = A\,x + B/x$, where
the value of A is assumed to remain constant throughout the process at a value of A=68. However, the value of B fluctuates slowly because of gradual environmental changes.

(a) If the process was recently measured in terms of (x,y) at (.5, 79), find the value of B at that instant.
(b) Using this value of B, determine the value of x that will minimize y. Assume y measures the production cost of monitoring. What is this value of y?
(c) This seems too unrealistic for you. You collect data over a 12 week period. Find the values of A and B that minimize the model:
$S = \Sigma\,(Y_i - (Ax_i + (B/x_i)))^2$

Week	1	2	3	4	5	6	7	8	9	10	11	12
Y	76.6	78	97.5	120.5	145	170.5	196	222	248	274	301	328
x	1.1	2.1	3	4.5	5	6	7.1	8	9	10	11.1	12

(d) If A and B are settings on a machine, then what settings would you use and why?

Part II.
The manufacture is planning the introduction of two new products, a 19-inch stereo color set with a manufacturer's suggested retail price (MSRP) of $339 and a 21-inch stereo color set with a MSRP of $399. The cost to the company is $195 per 19-inch set and $225 per 21-inch set, plus an additional $400,000 in fixed costs of initial parts, initial labor, and machinery. In a competitive market in which they desire to sell the sets, the number of sales per year will affect the average selling price. It is estimated that for each type of set, the average selling price drops by one cent for each additional unit sold. Furthermore, sales of 19-inch sets will affect the sales of 21-inch sets and vice-versa. it is estimated that the average selling price for the 19-inch set will be reduced by an additional 0.3 cents for each 21-inch set sold, and the price for the 21-inch set will decrease by 0.4 cents for each 19-inch set sold. We desire to provide them the optimal number of units of each type set to produce and to determine the expected profits. Recall Profit is revenue minus cost, P=R-C.

REQUIRED:

1. Formulate the model to maximize profits. Insure that you have accounted for all revenues and costs. Define all your variables.
2. Solve for the optimal levels of 21-inch and 19-inch sets to be manufactured using
(a) Classical Optimization-Calculus
(b) Combination of 3-D Surface Plot and Contour plot. (estimated answer is all right here)
(c) Newton-Raphson Method. Briefly explain why you can use this technique.

3. Using *Maple*, obtain a contour plot of the function. Color or in some other manner identify the optimal point. Illustrate the gradient search technique, starting from the initial point (0, 0). Only perform two or three iterations, showing the gradient, the distance traveled, and the new point. There is no requirement to obtain the optimal solution using this method.
4. Comment about the accuracy and rate of convergence (# iterations or difficulty) in obtaining a result in parts 2 and 3 above.
5. Comment on the solution in terms of the scenario. Did we use an appropriate techniques? Should there be any constraints that were not considered? Make your recommendation to the CEO concerns these TV sets.

PART II. Your company did well on their consulting job for the TV company. They have decided to hire you to do some further analysis.
Previously:
The manufacture is planning the introduction of two new products, a 19-inch stereo color set with a manufacturer's suggested retail price (MSRP) of $339 and a 21-inch stereo color set with a MSRP of $399. The cost to the company is $195 per 19-inch set and $225 per 21-inch set, plus an additional $400,000 in fixed costs of initial parts, initial labor, and machinery. In a competitive market in which they desire to sell the sets, the number of sales per year will affect the average selling price. It is estimated that for each type of set, the average selling price drops by one cent for each additional unit sold. Furthermore, sales of 19-inch sets will affect the sales of 21-inch sets and vice-versa. it is estimated that the average selling price for the 19-inch set will be reduced by an additional 0.3 cents for each 21-inch set sold, and the price for the 21-inch set will decrease by 0.4 cents for each 19-inch set sold. We desire to provide them the optimal number of units of each type set to produce and to determine the expected profits. Recall Profit is revenue minus cost, P=R-C.
Currently:
Above we assumed that the company has the potential to produce any number of TV sets per year. Now we realize that there is a limit on production capacity. Consideration of these two products came about because the company plans to discontinue manufacturing of its black-and-white sets, thus providing excess capacity at its assembly plants. This excess capacity could be sued to increase production of other existing product lines, but the company feels that these new products will be more profitable. It is estimated that the available production capacity will be sufficient to produce 10,000 sets per year (about 200 per week). The company has ample supply of 19-inch and 21-inch color tubes, chassis, and other standard components; however, circuit assemblies are in short supply. Also the 19-inch TV requires different circuit assemblies than the 21-inch TV. The supplier can deliver 8000 boards per year for the 21-inch model and 5000 boards per year for the 19-inch model. Taking this new information into account, what should the company now do?
Required:
1. Solve the above as a LaGrange Multiplier problem with equality constraints. Interpret the shadow prices.
2. Solve assuming the constraints are inequalities using KTC. Interpret the shadow prices.

OPTIMIZATION in MATHEMATICS

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We begin our discussion of optimization, modeling, and MAPLE with equality constrained problem from multivariable calculus. We present the graphical interpretation first with both contour plots and 3-D plots. Next, we illustrate the use of MAPLE to solve the set of partial derivatives leading to a solution. We interpret the solution in context of an application' example.

Graphical Interpretation

Maximize $z = -2x^2 + -y^2 + xy + 8x + 3y$
s.t. $3x + y = 6$

We obtained a contour plot of z from MAPLE and overlaid the single constraint onto the contour plot, see Figure 1. Let's see what information can we obtain from this graphical representation. First, we note that the unconstrained optimal does not lie on the constraint. We can estimate the unconstrained optimal $(x^*,y^*) = (2.3,1.3)$ as the approximate center of the inner most contour.. The optimal *constrained* solution lies at the point where the constraint is tangent to a contour of the function, *f*. This point (let's call it X*) is estimated to be about (1.8,1.0). Clearly, we see that the resource does not pass through the unconstrained maximum and thus, improvements can be made in the solution if we can afford more resource. Through more resource, the line will move until we get the line to pass through the unconstrained solution. At that point, we would no longer add (or subtract) any more resources (see Figure 2). We can gain valuable insights about the problem if we are able to plot the information and experiment with the constraints.

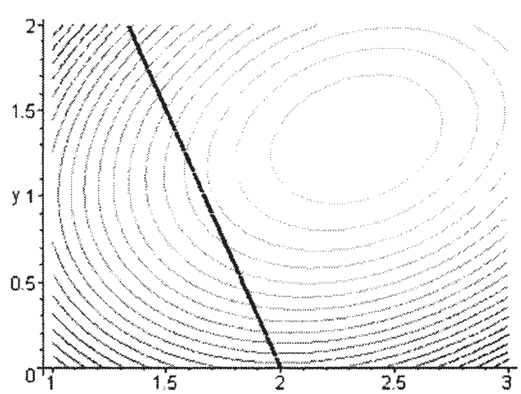

Figure 1. Contour plot of equality constraint example g(x)= 3x + y =6.

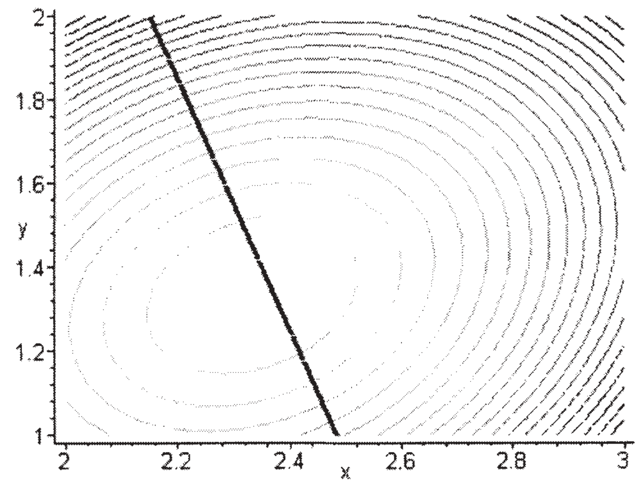

Figure 2. Adding more resource, $g(x) = 8.45 = 3x+y$

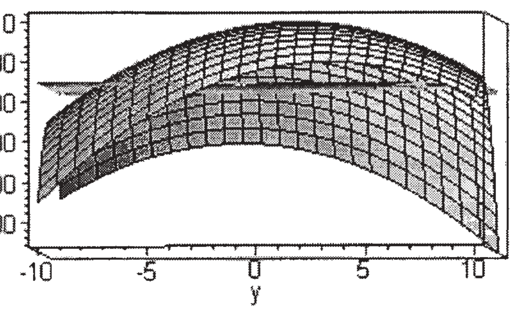

Figure 3. The 3-D plot

Computational Method of LaGrange Multipliers with Maple

You are employed as a consultant for a small oil transfer company. The management desires a minimum cost policy due to the restricted tank storage space. Historical records have been studied and a formula has been derived that describes system costs:

$$f(\mathbf{x}) = \sum_{n=1}^{N} (A_n B_n) / X_n + (H_n X_n) / 2$$

where:

A_n is the fixed costs for the nth item.

B_n is the withdrawal rate per unit time for the nth item.
H_n is the holding costs per unit time for the nth item.

The tank space constraint is given by:

$$g(\mathbf{x}) = \sum_{n=1}^{N} t_n X_n = T$$

where: t_n is the space required for the nth item (in correct units)
T is the available tank space (in correct units)

You determine the following information:

Item(n)	A_n ($)	B_n	H_n ($)	t_n (cubic feet)
1	9.6	3	0.47	1.4
2	4.27	5	0.26	2.62
3	6.42	4	0.61	1.71

You measure the storage tanks and find that there is only 22 cubic feet of space available. Find the optimal solution using a *minimum cost policy*.

First, solve the unconstrained problem.

If we assume that $\lambda = 0$, we first find an unconstrained optimal solution.

```
>Lf:=  (9.6)*(3)/x + .47*x/2 + (4.27)*(5)/y + .26*y/2 + (6.42)*(4)/z +
.61*z/2;
```

$$Lf := 28.8\frac{1}{x} + .2350000000\,x + \frac{21.35}{y} + .1300000000\,y + \frac{25.68}{z} + .3050000000\,z$$

```
>pd1:=diff(Lf,x);
```

$$pd1 := -28.8\frac{1}{x^2} + .2350000000$$

```
>pd2:=diff(Lf,y);
```

$$pd2 := -21.35\frac{1}{y^2} + .1300000000$$

```
>pd3:=diff(Lf,z);
```

$$pd3 := -25.68\frac{1}{z^2} + .3050000000$$

```
>solve({pd1=0,pd2=0,pd3=0},{x,y,z});
```

$\{z = -9.175877141, y = -12.81525533, x = -11.07037450\},$
$\{x = 11.07037450, z = -9.175877141, y = -12.81525533\},$
$\{z = 9.175877141, y = -12.81525533, x = -11.07037450\},$
$\{z = 9.175877141, x = 11.07037450, y = -12.81525533\},$
$\{y = 12.81525533, z = -9.175877141, x = -11.07037450\},$
$\{x = 11.07037450, y = 12.81525533, z = -9.175877141\},$
$\{z = 9.175877141, y = 12.81525533, x = -11.07037450\},$
$\{z = 9.175877141, x = 11.07037450, y = 12.81525533\}$

The only useful solution from the above solutions is where each of the variables *x, y,* and *z* are greater than or equal to zero.

x= 11.07037450, y= 12.81525533, z= 9.175877141

This unconstrained solution is (x*,y*,z*) = (11.07, 12.82, 9.176). This solution provides an upper bound since those values will not satisfy the constraint,
1.4 x+ 2.62 y + 1.71 z = 22.

The constrained solution is found by the following methodology:
1) Set up the LaGrangian function, L.
Let x = item 1
y= item 2
z= item 3

$L(x,y,z,\lambda) = (9.6)(3)/x + .47x/2 + (4.27)(5)/y + .26y/2 + (6.42)(4)/z + .61z/2 + \lambda\, [\, 1.4\, x + 2.62\, y + 1.71\, z - 22]$
2) Find all the partial derivatives set equal to zero.
$L_x = -28.8\, x^{-2} + .235 + 1.4\, \lambda = 0$
$L_y = -21.35\, y^{-2} + .13 + 2.62\, \lambda = 0$
$L_z = -25.68\, z^{-2} + .305 + 1.71\, \lambda = 0$
$L_\lambda = 1.4\, x + 2.62\, y + 1.71\, z - 22 = 0$

```
> L:=  (9.6)*(3)/x + .47*x/2 + (4.27)*(5)/y + .26*y/2 + (6.42)*(4)/z +
.61*z/2 + l1 *( 1.4* x + 2.62* y + 1.71* z -22);
```

$$L := 28.8 \frac{1}{x} + .2350000000\, x + \frac{21.35}{y} + .1300000000\, y + \frac{25.68}{z} + .3050000000\, z + l1\, (1.4\, x + 2.62\, y + 1.71\, z - 22)$$

```
> nc:=grad(L, [x,y,z,l1]);
```

$$nc := \left[-28.8 \frac{1}{x^2} + .2350000000 + 1.4\, l1, -21.35 \frac{1}{y^2} + .1300000000 + 2.62\, l1, -25.68 \frac{1}{z^2} + .3050000000 + 1.71\, l1, 1.4\, x + 2.62\, y + 1.71\, z - 22 \right]$$

3) Solve the set of partial derivatives .

```
> solve({-28.8*1/(x^2)+.2350000000+1.4*l1, -
21.35*1/(y^2)+.1300000000+2.62*l1, -25.68*1/(z^2)+.3050000000+1.71*l1,
1.4*x+2.62*y+1.71*z-22},{x,y,z,l1});
```

$\{y = 3.213131453, z = 4.044536153, x = 4.761027695, l1 = .7396771332\}$

```
> subs({y = 3.213131453, z = 4.044536153, x = 4.761027695, l1 =
.7396771332},L);
```

$$21.81316118$$

4) Check to see if we found a maximum or minimum value.
Do we have the minimum?
The Hessian matrix, H is:

```
> h:=hessian(L, [x,y,z]);
>
```

$$h := \begin{bmatrix} 57.6\frac{1}{x^3} & 0 & 0 \\ 0 & 42.70\frac{1}{y^3} & 0 \\ 0 & 0 & 51.36\frac{1}{z^3} \end{bmatrix}$$

```
> h1:=det(h);
```

$$h1 := 126320.9472\frac{1}{x^3\,y^3\,z^3}$$

```
> subs({y = 3.213131453, z = 4.044536153, x = 4.761027695, l1 =
.7396771332},h1);
```

$$.5333123053$$

The Hessian Matrix is positive definite at (x*,y*,z*)= (4.5, 3, 3.82). Therefore, the solution found is the minimum for this convex function with a linear equality constraint.
5) Interpret the shadow price, λ.

Should we add storage space? We know from the unconstrained solution that, if possible, we would add storage space to decrease the costs. Additionally, we have found the value of λ was 0.85, which suggests that any small increase (Δ)in the RHS of the constraint causes the objective function to decrease by approximately .85Δ. Thus, the total cost of adding an extra storage tank would have to be less than the savings incurred by adding the new tank.