EXPANDING CURVES AND ARC LENGTH

Dr. Joel C. Fowler
Southern Polytechnic State University
1100 South Marietta Parkway
Marietta, GA 30060
jfowler@spsu.edu

A classic puzzle assumes that the earth is a perfect sphere and a steel band is stretched around the equator. If six feet is added to the band, which is adjusted to a uniform height above the earth all the way around the equator, how large of an object could be slipped under the adjusted band, a playing card, a marble, a baseball, or a basketball? Since the ratio a circle's radius to its circumference is 2π, the answer is that the band will be over 11 inches off of the surface and hence a basketball will fit under. A circle satisfies $2\pi\,\Delta r = \Delta C$, where r is the radius and C is the circumference. A natural question is how this relationship might change for other curves. Let C be a plane curve defined by the parametrization $C = \{ <x(t),\ y(t)>:\ a \le t \le b \}$. The expanded curve, C_e, for a positive real number e, is the curve obtained by traveling e from C along a normal. That is:

$$C_e = \left\{ <x(t),\ y(t)> + e\left(\frac{<y'(t),\ -x'(t)>}{\sqrt{x'(t)^2+y'(t)^2}} \right) :\ a \le t \le b \right\} = \{ <x_e(t),\ y_e(t)>:\ a \le t \le b \}$$

We have chosen the normal that points outward for a counterclockwise traversal of a circle. Depending on $<x(t),\ y(t)>$, this could, however, represent an inward pointing normal and a "contraction". The change in arc length from C to C_e is then given by

$$\Delta L = \int_a^b \left(\sqrt{x_e'(t)^2+y_e'(t)^2} - \sqrt{x'(t)^2+y'(t)^2} \right) dt .$$

Evaluation of arc length integrals by hand is normally difficult, except for carefully chosen curves. Maple makes graphing and approximating arc length for various curves relatively easy. The following curves illustrate a variety of behavior.

Example 1: An ellipse. Let $C = \{ <2\cos(t),\ 7\sin(t)> : 0 \le t \le 2\pi \}$. Using Maple we find that, for all tested values of e, $\Delta L/e \approx 2\pi$.

Example 2: A segment of a parabola. Let $C = \{ <t,\ t^2> : -2 \le t \le 2 \}$. For this curve we find that, for all tested values of e, $\Delta L/e \approx 2.651635328$.

Example 3: A limacon. Let C = $\{<\cos(\theta)(7+6\sin(\theta)),\ \sin(\theta)(7+6\sin(\theta))>: 0 \le \theta \le 2\pi\}$. We find that, although $\Delta L/e \approx 2\pi$ for small values of e, as e increases $\Delta L/e$ appears to increase. Once graphed we see that the expanded curve exhibits doubling back for larger values of e when the normal points "inward" at the dimple.

Example 4: A cycloid. Let C = $\{<\pi-\theta+\sin(\theta),\ 1-\cos(\theta)>: 0 \le \theta \le 2\pi\}$. Using Maple we find that, regardless of the value of e, $\Delta L/e \approx \pi$.

Example 5: A 4-leaf rose. Let C = $\{<\cos(\theta)(\sin(2\theta)),\ \sin(\theta)(\sin(2\theta))>: 0 \le \theta \le 2\pi\}$.

We find that, for all tested values of e, $\Delta L/e \approx 18.84955592 \approx 6\pi$.

Example 6: Infinity. Let C = $\{<\sin(t),\ \sin(2t)>: 0 \le t \le 2\pi\}$. We find that, for small values of e, $\Delta L/e \approx 0$. Larger values result in the doubling back behavior we saw in Example 3, and a non-zero value.

Example 7: A sine wave. Let C = $\{<t,\ \sin(t)>: 0 \le t \le 2\pi\}$. For values of e<1, $\Delta L/e \approx 0$. e>1 results in the doubling back we saw earlier, and a non-zero $\Delta L/e$.

We can see that, while $\Delta L/e$ isn't completely random, it varies with some properties of the curve. If the normal is pointing "outward", then $\Delta L/e$ doesn't vary with e. If the normal points inward, then, for large e, there is "doubling back" and $\Delta L/e$ does vary with e. $\Delta L/e$ doesn't vary for small values of e.

We now look at some general results that help to explain the behavior we have just seen. Although using vectors from a parametrization is the most general approach, we will first look at the special case of a function y = f(x), since the ideas involved are accessible at a lower level. The reader is encouraged to return to the appropriate examples following each theorem to see how each explains the experimental behavior found via Maple. By applying the theorems one can also determine how a more general class of curves would behave.

Curves Defined by Functions

We can parametrize a segment of a function as $C = \{<t,\ f(t)>:\ a \le t \le b\}$. Then

$C_e = \left\{ <t,\ f(t)> + e\left(\frac{<f'(t), -1>}{\sqrt{1+f'(t)^2}} \right) : a \le t \le b \right\}$. After some computation (which can be done with Maple or by hand) we then find that:

$$x_e'(t) = 1 + e\frac{f''(t)}{(1+f'(t)^2)^{3/2}} \quad \text{and} \quad y_e'(t) = f'(t) + e\frac{f'(t)f''(t)}{(1+f'(t)^2)^{3/2}} .$$

Note that $f'(t) = y_e'(t)/x_e'(t)$. This observation helps in evaluating:

$$\Delta L = \int_a^b \left(\sqrt{x_e'(t)^2 + y_e'(t)^2} - \sqrt{x'(t)^2 + y'(t)^2} \right) dt$$

$$= \int_a^b \left(\sqrt{x_e'(t)^2 + (x_e'(t)\, f'(t))^2} - \sqrt{1+f'(t)^2} \right) dt = \int_a^b \left((\,|x_e'(t)| - 1)\sqrt{1+f'(t)^2} \right) dt .$$

Checking the sign of $x_e'(t)$ and noting that $\int \frac{f''(t)}{1+f'(t)^2}\, dt = \tan^{-1}(f'(x)) + C$ yields:

Theorem 1: Let $y = f(x)$ and its first and second derivatives be continuous on [a,b]. Assume that for all $x \in [a, b]$, $f''(x) \ge 0$ or $e \le -\frac{(1+f'(x)^2)^{3/2}}{f''(x)}$. Then

$$\Delta L/e = \tan^{-1}(f'(b)) - \tan^{-1}(f'(a)) .$$

The conditions for $x_e'(t) \ge 0$ are natural. $f''(x) \ge 0$ implies the curve is concave up so the normal points out. If $f''(x) < 0$ the normal points in. For $f''(x) < 0$ $e \le -(1 + f'(x)^2)^{3/2} / f''(x)$ requires that e < the radius of curvature. Noting $\tan^{-1}(f'(x))$ = the tangent angle to f at x leads to an alternate expression of Theorem 1:

Theorem 1: Let $y = f(x)$ and its first and second derivatives be continuous on [a,b]. If e is less than the radius of curvature at all points of downward concavity in [a,b], then $\Delta L / e$ is the change in the angle of the tangent to the curve from x = a to x = b.

For curves with vertical tangents we may use improper integral techniques to show:

Theorem 2: Let $y = f(x)$ and its first and second derivatives be continuous on (a,b), f be vertical at a and b, and for all $x \in [a, b]$, $f''(x) \geq 0$ or $e \leq -(1 + f'(x)^2)^{3/2}/f''(x)$. Then $\Delta L/e$ depends only on the concavity of f at the endpoints. Upward concavity near an endpoint contributes $+\pi/2$ to $\Delta L/e$, while downward concavity near an endpoint contributes $-\pi/2$ to $\Delta L/e$.

Curves Defined by General Parametrizations

More generally let $C = \{ <x(t),\ y(t)>:\ a \leq t \leq b \}$ and

$$C_e = \left\{ <x(t),\ y(t)> + e\left(\frac{<y'(t),\ -x'(t)>}{\sqrt{x'(t)^2 + y'(t)^2}} \right) :\ a \leq t \leq b \right\} = \{ <x_e(t),\ y_e(t)>:\ a \leq t \leq b \} .$$

After some computation (which can be done with Maple or by hand) we then find that:

$$\Delta L/e = \int_a^b ((\,|\,M(t)\,|\, - 1)\sqrt{x'(t)^2 + y'(t)^2}\ )\,dt \quad \text{where}$$

$M(t) = 1 + e\,(x'(t)y''(t) - x''(t)y'(t))/(x'(t)^2 + y'(t)^2)^{3/2}$. Examining M(t) and noting

$d(\tan^{-1}(y'(t)/x'(t)))/dt = (x'(t)y''(t) - x''(t)y'(t))/(x'(t)^2 + y'(t)^2)$ yields:

Theorem 3: Let x(t) and y(t) and their first and second derivatives be continuous on [a,b]. Assume x'(t) is nonzero on [a,b]. If e is less than the radius of curvature at all points in [a,b] at which the tangent slope is decreasing, then $\Delta L/e$ is the change in the angle of the tangent to the curve from t = a to t = b.

This theorem can be modified using improper integral techniques to handle curves for which x'(t) is zero at finitely many points of [a,b].

Theorem 4: Let x(t) and y(t) and their first and second derivatives be continuous on [a,b]. Assume that the curve is vertical at only finitely many points of [a, b] and that e is less than the radius of curvature at all points in [a,b] at which the tangent slope is decreasing. Then $\Delta L/e$ is the sum of the following:

1) + or - the angle of the tangent to the curve at any non-vertical endpoints, + at b, - at a.

2) $+\pi/2$ for every one-sided neighborhood of a point of vertical tangency in which the slope increases.

3) $-\pi/2$ every one-sided neighborhood of a point of vertical tangency in which the slope

decreases.

Vector Considerations

Many of these ideas can be viewed from the standpoint of vector calculus. Let C be defined by the position vector $\vec{r}(t) = <x(t),\ y(t)>$. Then $\vec{T}(t) = \dfrac{\vec{r}^{\,\prime}(t)}{\|\vec{r}^{\,\prime}(t)\|} = \dfrac{<x'(t),\ y'(t)>}{\sqrt{x'(t)^2 + y'(t)^2}}$ and

$\vec{N}(t) = \dfrac{<y'(t),\ -x'(t)>}{\sqrt{x'(t)^2 + y'(t)^2}}$ are tangent and normal vectors. The expanded curve is given by the position vector $\vec{r}_e(t) = \vec{r}(t) + e\,\vec{N}(t)$ and

$$\Delta L = \int_a^b (\|\vec{r}_e^{\,\prime}(t)\| - \|\vec{r}^{\,\prime}(t)\|)\, dt \qquad = \int_a^b (\|\vec{r}^{\,\prime}(t) + e\vec{N}^{\prime}(t)\| - \|\vec{r}^{\,\prime}(t)\|)\, dt$$

Note that $\vec{N}(t)$ is a unit vector, and hence is orthogonal to $\vec{N}^{\prime}(t)$ and also $\vec{T}(t)$. Hence $\vec{N}^{\prime}(t)$ and $\vec{r}^{\,\prime}(t)$ are parallel and $\vec{N}^{\prime}(t) = S(t)\,\vec{r}^{\,\prime}(t)$, where $S(t) = \dfrac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}$. Hence

$$\Delta L = \int_a^b (\|\vec{r}^{\,\prime}(t) + e\,S(t)\,\vec{r}^{\,\prime}(t)\| - \|\vec{r}^{\,\prime}(t)\|)\, dt = \int_a^b (|\ 1 + e\,S(t)\ | - 1)\, \|\vec{r}^{\,\prime}(t)\|\ dt\ .$$

For non-negative $1 + e\,S(t)$, $\Delta L = \int_a^b |\ e\,S(t)\ |\ \|\vec{r}^{\,\prime}(t)\|\ dt = e\int_a^b \|\vec{N}^{\prime}(t)\|\, dt$ and we have the invariant $\Delta L/e = \int_a^b \|\vec{N}^{\prime}(t)\|\, dt$. Since $\|\vec{N}^{\prime}(t)\|$ is the rate of change of the tangent angle, when the integral is not improper, $\Delta L/e$ is the angle the tangent turns through.

Further Problems

One could investigate particular classes of curves, such as those presented in polar form. We have not discussed when C_e is discontinuous, which is not hard to produce examples of. We have also not developed the case in which $\Delta L/e$ is not invariant. For example, is there a limiting value as e goes to infinity? Another direction to take is to generalize these ideas to three dimensions. One could examine how the arc length changes for curves expanded along the principal normal. Another question is how surface area changes when a surface is generated by following the principal normal to produce another surface.