THE INSOLVABILITY OF THE QUINTIC AND ITS IMPLICATION TO TECHNOLOGY

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I. A polynomial equation of degree n is of the form:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

where the coefficients a_i are in general complex. We are familiar with the solutions when the degree is 2 -- we have the quadratic formula:

$$x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}.$$

We also have the formulas for the cubic (Tartaglia) and the quartic equations (Ferrari, Descartes). But after hundreds of years of trying, mathematicians have not found a solution to the quintic equation. Finally, Ruffini, in 1799, and Abel, in 1824, proved that no such formula exists for the general quintic polynomial. But both proofs had some gaps. In 1832, Galois was able to determine precisely those polynomials whose roots can be found by a formula involving the nth roots of numbers and the usual field operations of addition, subtraction, multiplication, and division. In so doing, he founded the Theory of Groups.

II. The proof of the insolvability of the quintic is provided only by group theory, which is a totally different level now than the proofs for the classical equations of degree 1 to 4. The criterion for whether a polynomial equation can be solved by radicals or not is determined by what is now called solvable groups. The formula sought is one which involves the coefficients a_i of the polynomial and the operations addition, subtraction, multiplication, division, and a finite number of extraction of roots. If the roots of $f(x)$ can be obtained by such a formula, we say $f(x)$ is *solvable by radicals.*

From the Fundamental Theorem of Algebra we know that an nth degree polynomial with complex coefficients has n complex roots. Let F be the 'smallest' field of complex numbers containing the coefficients a_i of $f(x)$. This means that if H is a field containing the coefficients a_i, then $F \subseteq H$. Let E be the smallest field containing F and the roots of $f(x)$. Now the set of automorphisms of E forms a group under the composition of mappings. The automorphisms of E which map every element f of F onto itself is a subgroup G of the group of all automorphisms of E. This group is called the Galois group of the polynomial $f(x)$, in honor of its founder. In 1832, Galois proved (essentially) the following theorem: An equation $f(x) = 0$ is solvable by radicals if and only if the Galois group of $f(x)$ is solvable. But the general equation of degree 5 is isomorphic to the symmetric group S_5. And the symmetric group S_5 is not a solvable group. Thus, it turns out that not all equations of degree 5 are solvable by radicals.

III. Although the general quintic equation is not solvable in closed form, we can find its roots by numerical means. (1) From the theory of equations, we know that an equation of odd degree has a real solution. This is clear from the intermediate value theorem: If f is a polynomial with real coefficients and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one value c between a and b for which $f(c) = 0$. Thus, with c known, we can factor this root out and obtain all the other roots from the quartic formula. (2) One method for approximating the root of a quintic is Newton's method, a standard technique in calculus, which works for a general function (not necessarily polynomial) f.

There are other numerical methods for finding the roots of a quintic not using radicals. In 1827, Abel and Jacobi developed the theta functions. In 1858, Hermite, Kronecker, and Brioschi independently showed that any quintic could be solved by elliptic modular functions derived from these theta functions. These are defined as infinite sums:

$$\phi(z) = \sqrt[3]{\frac{\theta_2(0,z)^4}{\theta_3(0,z)^4}}\text{ , elliptic modular function}$$

where the theta functions are:

$$\theta_2(z,q) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos(2n+1)z$$

$$\theta_3(z,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz).$$

The solutions of any polynomial equation can also be expressed in terms of the generalized hypergeometric function – a quotient of general products of series of powers:

$${}_pF_q(a_1,\ldots,a_p : b_1,\ldots,b_q : z) = \sum_{k=0}^{\infty} \frac{\Pi_{i=1}^{p}(a_i)_k z^k}{\Pi_{i=1}^{q}(b_i)_k k!}$$

where the 'Pochhammer symbol' $(a_i)_k$ is defined by:

$$(a_i)_k = \prod_{j=1}^{k}(a_i + j - 1).$$

Brief summaries of these functions are found at http://library.wolfram.com/examples/ quintic/theta.html and at http://library.wolfram.com/examples/quintic/hypergeo.html.

IV. Solutions to the quadratic equation have been known since ancient times. Solutions to the cubic and the quartic equations were discovered in the renaissance, the first noteworthy results not ascribed to the ancient mathematicians. The quintic equation defied the best minds for 300 years; it was solved only in the 1830s by showing that it is in general insolvable by radicals. Group theory was also successful in showing that all higher-degree polynomial equations are insolvable as well. Recent research shows that any algebraic equation can be solved by modular functions. In fact, numerical solutions exist for any algebraic equation.

V. Bibliography

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