MODELING WITH DISCRETE DYNAMICAL SYSTEMS

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Using a Discrete Dynamical Systems (DDS) involves modeling with the paradigm:

Future = Present + Change

A DDS is a discrete function that can be used to model many situations, such as

Mortgage of a home
Car financing
Investment or Financial alternatives
Drug dosages
Population dynamics
Genetics

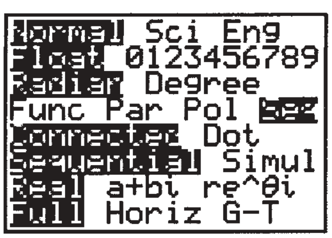
So, let's do a quick example.

Modeling One Time Prescribed Drug Dosages

Novocain (Procainamide Hydrochloride), injected as an anesthetic for minor surgical and dental procedures, is eliminated from the body primarily by the kidneys. Loosely speaking, during any 1-hour period, the kidneys take a fixed percentage of the blood and remove medicine from the blood. Let's assume the kidneys purify 1/5 of the blood every one-hour period.

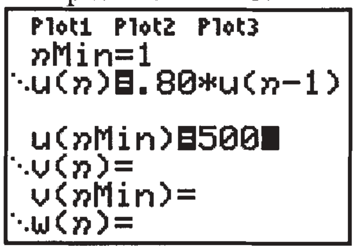
Let: $u(n)$ = the amount of the prescribed drug in our system after n (one-hour) periods.
$u(n+1) = u(n) - .20\,u(n) = .80\,u(n)$, $n = 0, 1, 2, 3, \ldots$
Let's assume we take 500 mg of the drug in period 0, so $u(0)=500$.
Let's use the calculator to iterate and graph the DDS to see what happens over a long period of time. We can build the solution on the TI-83 Plus.

1. Go to **MODE**, func, **SEQ. SEQUENTIAL** (2^{nd} quit)



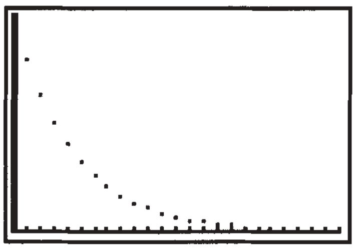
2. Go to y=

and put in our DDS:



3. Set the window for n [0,24], x [0,24], y[0,505]

4. Press **Graph**



We clearly see that this drug decays over time. There is less and less of the drug in our system after each hour. Remember that tingly feeling as the feeling begins to return to your face.

Now, let's consider a drug taken more often than once. CIPRO is a drug for combating many infections, including anthrax. Let's assume that during a one-hour period that our kidneys purify 1/4 of this drug from the blood. Let's assume that the dosage is 16 mg each time period. So our model is,

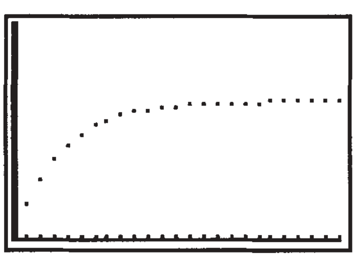
d(n+1)= .75 d (n), d(0)=1 6 mg, n=0,1,2,3,... (in one-hr periods)

Let's assume that to be effective, you must have at least 6.75 mg of the medicine in your blood. How often should you take the medicine?

Let's remodel assuming that every 4-hour period we take 16 mg of the drug and that we don't have any in our system when we begin, *d(0)=0.*

The model is *d(n+1) = .75 * d(n) + 16* for *n=0,1,2,3,...*(*n* now represents 4-hour periods).

What happens now? Look at a long period of use for this drug. Describe what you see from the graphical output.



We can also look at the Tabular output:

n	u(n)	
40	63.999	
41	64	
42	64	
43	64	
44	64	
45	64	
46	64	

n=46

This shows a ***stable equilibrium*** value. The definition of an equilibrium value is when *d(n+1)=d(n), INPUT=OUTPUT,* and there is no change. This equilibrium value is 64 mg of this drug. Stable is when the DDS always moves toward the equilibrium value if it does not start at that value.

D(n+1)=0.75*D(n)+16, D(0) = 0

Note: the notational difference between the calculator and our form.

We need to enter:

*u(n)=.75*u(n-1)=16, u(0)=0*

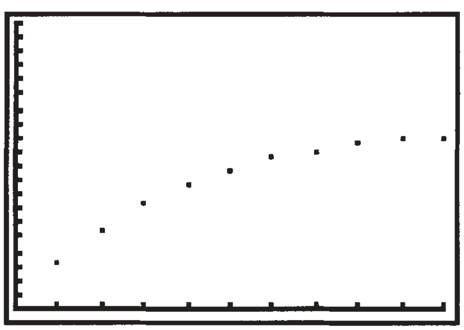
nMin is the start for the counter n

u(nMin) = u(0) ---the initial condition.

```
Plot1 Plot2 Plot3
 nMin=0
·.u(n)■.75*u(n-1)
+16
 u(nMin)■0■
·.v(n)=
 v(nMin)=
·.w(n)=
```

Set WINDOW for *n* [0,15], *x* [0,15], *y* [0,100].

Press Graph



Press 2^{nd} Table and scroll for positive values of *n*. What happens as *n* gets large? Where is the system, *u(n)*, going? Is there an equilibrium value? Is it stable? (There is that word again. What does that mean?)

Nonlinear Discrete Dynamical Systems & Systems of DDS

We often model population growth by assuming that the change in population is directly proportional to the current size of the given population. It might appear reasonable at first examination but the long-term behavior of growth without bound is disturbing. Why would growth without bound of a yeast culture in a jar (or controlled space) be alarming?

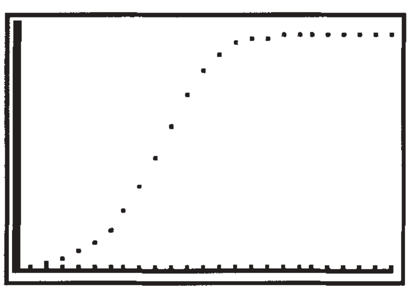
There are certain factors that affect population growth, such as resources like food, oxygen, and space. These resources can support some maximum population. As this number is approached, the change (or growth rate) should decrease and the population should never exceed its resource supported amount.

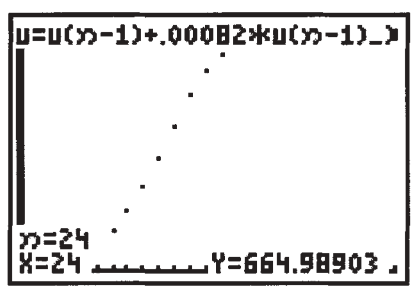
Problem Identification: Predict the growth of yeast in a controlled environment as a function of the resources available and the current population.

Assumptions and Variables: We assume that the population size is best described by the weight of the biomass of the culture. We define y(n) as the population size of the yeast culture after period n. There exists a maximum carry capacity, M, that is sustainable by the resources available. The yeast culture is growing under the conditions established.

Model:
$y(n+1) = y(n) + k\,y(n)\,(M-y(n))$ where
$y(n)$ is the population size after n hours
n is the number of hours
k is the growth rate of the culture
M is the carrying capacity of our system

In our experiment, we find by data collection that the growth rate, k, is approximately 0.00082 and the carrying capacity of biomass is 665. This model is $y(n+1) = y(n) + .00082\,y(n)\,(665-y(n))$. Again, this is nonlinear because of the $y^2(n)$ term. The solution iterated on the Ti-83 Plus from an initial condition, biomass, of 9.6 is





The model shows stability in that the population (biomass) of the yeast culture approaches 665 as n gets large. Thus, the population is eventually stable at approximately 665 units.

Consider the model a(n+1) = r a(n) (1-a(n)). Let a(0)=0.2. Determine the numerical and graphical solution for the following values of r. Find the patterns in the solution. One of these patterns exhibits chaotic behavior. r= 2, r=3, r=3.6, r=3.7

Systems of DDS

Now, we consider systems of difference equations (DDS). For selected initial conditions, we build numerical solutions to get an sense of long term behavior of the system. For the systems that we will study, we will find their equilibrium values. We then explore starting values near the equilibrium values to see if start close to an equilibrium value, will the system remain close, approach the equilibrium value, or not remain close

Merchants located Downtown and in Malls

Let's consider the attempt to revitalize the downtown section of a small city with merchants. There are merchants downtown and other in the large mall. Suppose historical records determined that 60% of the downtown merchants remain downtown, while 40% move to the mall. We find the 70% of the mall merchants want to remain in the mall, but 30% want to move to downtown. Build a model to determine the long-term behavior of these merchants based upon this historical data.

D(n) = the number of merchants operating downtown at the end of n months

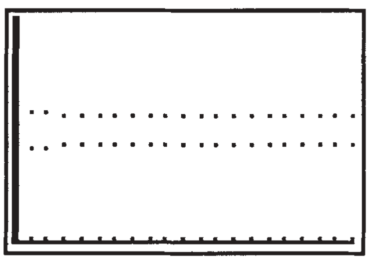
M(n) = the number of merchants operating at the mall at the end of n months

<u>The Model</u>:

$D(n+1) = .60\, D(n) + .3\, M(n)$

$M(n+1) = .4\, D(n) + .7\, M(n)$

There are initially 150 merchants in the mall and 100 downtown. The long-term behavior



is found by evaluating numerically these equations, and with a table,

n	u(n)	v(n)
15	107.14	142.86
16	107.14	142.86
17	107.14	142.86
18	107.14	142.86
19	107.14	142.86
20	107.14	142.86
21	107.14	142.86

n=15

The long-term behavior shows that eventually (without other influences) that of the 250 merchants about 107 merchants will be downtown and 143 will be in the mall. As a result of our research, we might want to try to attract new businesses to the community and add incentives for operating the business downtown.