IMPROVING LINEAR ALGEBRA INSTRUCTION USING WEBWORK AND THE ATLAST PROBLEMS

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Introduction: Linear Algebra is a central course in the mathematics curriculum. It is a bridge course between the calculus curriculum and the upper-level mathematics curriculum and often serves a diversity of audiences. In addition to mathematics majors and preservice mathematics teachers, there are many students from computer science, engineering, physics, biology, and even business enrolled in the course. Finding more effective ways of teaching Linear Algebra is important for enabling all these students to fulfilling their potential in their future careers.

The last decade has seen the development of many innovative tools for improving Linear Algebra instruction. With support from NSF-DUE grant #0126928, we at the College of New Jersey have integrated the use of two of these tools, WeBWorK and the ATLAST Linear Algebra problems, into our Linear Algebra course with great success. WeBWorK is a computerized mathematics homework system over the internet that delivers instant feedback to students. The ATLAST problems are a collection of innovative Linear Algebra problems that emphasize visualization and real-world applications. Through the use of both tools, our linear algebra teaching has become more effective. Students have a better understanding of the material; they acquire this understanding more quickly and easily than in the past; and perhaps most importantly, they are given more motivation for learning Linear Algebra.

Project Goals at TCNJ: Over the last two years, we have transformed the linear algebra curriculum at TCNJ by the introduction of the ATLAST and WeBWorK software tools. While successful, an immediate result has been a significant increase in the amount of student homework that must be graded. At first glance, this seems contradictory as the WeBWorK computerized grading system is meant to efficiently grade homework. WeBWorK certainly does this, but only for the linear algebra problems that have been created for the WeBWorK system. The WeBWorK linear algebra library (included with the standard WeBWorK installation) is a comprehensive set of problems, similar to those found in a standard textbook, covering all aspects of the typical linear algebra curriculum. WeBWorK effectively eliminates the time needed to grade these homework problems.

On the other hand, the ATLAST linear algebra problems are not part of the current WeBWorK problem library. This is to be expected as the ATLAST problems were originally developed as a supplement to the problems found in the textbooks. With their stress on real-world applications and developing student's visualization skills, the ATLAST problems should be a part of every linear algebra course. Unfortunately, they are time-consuming to grade. The current goal of our work at TCNJ is to remove this liability by adapting ATLAST-type problems to be a part of the WeBWorK computerized grading system. By providing a more efficient way to grade these problems, we hope to enable more instructors nationwide to make use of these educational tools.

History of WeBWorK: WeBWorK is a widely-used NSF-funded computerized homework grading system that provides instant feedback to students (information can be found at webwork.math.rochester.edu). It was developed over the last nine years at the University of Rochester by Michael Gage and Arnold Pizer and is a significant improvement upon the vast majority of automatic grading systems. In recognition of its strengths, it was awarded the 1999 ICTCM Mathematics Award for Excellence and Innovation with. In addition to true/false and multiple-choice questions, WeBWorK can grade homework questions whose answers are quite complicated mathematical expressions. For example, WeBWorK is used to grade derivative and integral problems in calculus whose answer is a single variable function. WeBWorK is also powerful enough to be able to recognize mathematically equivalent answers that may differ superficially. Originally designed for Calculus, the virtues of WeBWorK make it ideal for any mathematics course with computational homework problems.

WeBWorK has been shown to be effective in improving mathematics instruction. Chuck Weibel and Lew Hirsch at Rutgers University (New Brunswick, NJ) conducted a controlled experiment on the use of WeBWorK in Rutgers' calculus course for non-science majors and found that students who did at least 80% of the WeBWorK exercises performed a full letter grade better than students who did fewer than 50% of the WeBWorK problems (4). Such a finding is to be expected as educational research has shown that students learn more effectively and efficiently when they have a solid understanding of previously taught material. The use of WeBWorK eliminates the delay involved in traditional homework, between when homework is turned in and when it is returned. It also increases a student's ability to learn from their mistakes while doing their homework.

History of the ATLAST Project: The ATLAST (Augmenting the Teaching and Learning of Linear Algebra through Software Tools) project was an NSF-DUE funded initiative from 1992 to 1997 designed to implement the recommendations of the Linear Algebra Curriculum Study Group (LACSG). LACSG (1) recommended that linear algebra instruction would be improved by teaching the first course in Linear Algebra from a matrix-oriented viewpoint and by using computer software to help students visualize and master the material. The ATLAST project held 18 workshops involving 425 faculty and resulted in the development of a comprehensive library of linear algebra problems and computer modules for linear algebra (2).

Why ATLAST? The goals of the ATLAST problems and computer modules are to develop understanding of linear algebra concepts from a different angle than that provided by the exercises traditionally found in the textbooks. Some exercises are aimed at developing students' visualization skills through the use of computer modules such as java applets, while other exercises present applications of linear algebra to real-world mathematical problems. A common theme of all the problems is to be innovative in the presentation and use of linear algebra.

One of the strengths of the ATLAST problems is the use of *challenging* problems that force students to think. The use of a real-world problem often means that a student must work in a high dimensional vector space. To work through these problems without computational difficulties, students use the computer algebra system MatLab (additional Linear Algebra exercises using MatLab can be found in (3)).

Our ATLAST Results: The results have been striking. The use of MatLab enables students to work through difficult problems without computational difficulties, enabling students to deepen their conceptual understanding. The more realistic linear algebra problems that MatLab makes possible (realistic problems tend to have a computational difficulty that prevents them from being used in a course that does not use MatLab) improve student motivation by allowing students to see the applications of Linear Algebra. We have noticed this in particular for the large numbers of computer science majors in our classes.

An Example of an ATLAST Problem: As an illustration, we present an example of an ATLAST problem illustrating the theme of presenting a realistic application of linear algebra. As written, the problem forces deeper student understanding by forcing a student to explain their reasoning. By adapting the ATLAST problems as WeBWorK problems, we hope to eliminate the need for an instructor to grade the computational aspects of the problem; thereby, freeing up more time for the written explanations to be evaluated.

Markov Chains – A Voting Trends Example

A study has shown that, in a certain town, 90% of the people who vote for Party A in one election will vote the same way in the next election, and the remaining 10% will vote for Party B. The study also concluded that 80% of those voting for Party B will vote the same way in the next election, and the remaining 20% will vote for Party A.

In this example, there are two possible outcomes at each stage, a vote for Party A or a vote for Party B. The probability of either depends on how one voted at the previous stage. The four probabilities can be summarized in tabular form as follows:

Party A	Party B	
0.90	0.20	Party A
0.10	0.80	Party B

The first row of the table gives the probabilities that someone who last voted for Party A or Party B will vote for Party A in the next election. The second row gives the probabilities that the person will vote for Party B. The 2 x 2 matrix

$$A=\begin{pmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{pmatrix}$$

is called the *transition matrix* for the process. The entries of each column vector of A are probabilities that add up to 1. Such vectors are called *probability vectors.* A square matrix is said to be *stochastic* if each of its column vectors is a probability vector.

If, initially, there were 5000 votes for Party A and 5000 votes for Party B, then to predict the voting in the next election, set

$$x_0 = \begin{pmatrix} 5000 \\ 5000 \end{pmatrix}, \quad x_1 = Ax_0 = \begin{pmatrix} 5500 \\ 4500 \end{pmatrix}$$

One can predict future election results by setting $x_{n+1} = Ax_n$ for $n = 1, 2, \ldots$ The vectors x_i produced in this manner are referred to as *state vectors*, and the sequence of state vectors is called a *Markov chain.* If one divides the entries of the initial state vector x_0 by 10,000 (the total population), the entries of the new vector will represent the proportions of the population in each category. Thus, if we take

$$x_0 = \begin{pmatrix} 0.50 \\ 0.50 \end{pmatrix}$$

then x_0 is a probability vector and it is easily seen that each of the state vectors in the resulting chain will also be probability vectors.

Consider the Markov chain for the voting trends example with initial state $x_0 = (0.50 \quad 0.50)^T$.

(a) Compute x_1, x_2, x_3 and explain the significance of the entries of each of these vectors. Compute also A^2x_0 and A^3x_0. How do these vectors compare to x_2 and x_3? Explain.

(b) In order to see the long range voting trends, compute the vectors $x_5, x_{10}, x_{15}, x_{20}, x_{25}$. Does the sequence of state vectors appear to be converging? In general, if the sequence of state vectors converges to a limit vector x, then x is said to be a *steady-state vector* for the Markov process.

(c) For each of the following initial vectors, determine whether or not the Markov chain will converge to a steady-state vector. For those that converge, how are the steady-state vectors related?

(i) $x_0 = \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$ (ii) $x_0 = \begin{pmatrix} 1.00 \\ 0.00 \end{pmatrix}$ (iii) $x_0 = \begin{pmatrix} 0.00 \\ 1.00 \end{pmatrix}$

(d) A steady-state vector will have the property that $Ax = x$. Verify that this is indeed the case for any steady-state vectors you have found. Interpret this result in terms of eigenvalues and eigenvectors.

(e) If the intitial vector is a probability vector, must the steady state vector also be a probability vector? Explain. Compute the eigenvalues and eigenvectors of the transition matrix A. In view of part (c), how many steady-state vectors are possible?

References

(1) David Carlson, Charles Johnson, David Lay, Duane Porter. The Linear Algebra Curriculum Study Group Recommendations for the First Course in Linear Algebra. College Mathematics Journal, 24:1 (1993) 41-46.

(2) ATLAST Computer Exercises for Linear Algebra, 2nd ed., Edited by Stephen Leon, Eugene Herman, Richard Faulkenberry, Prentice-Hall, 2003.

(3) Linear Algebra Labs with MatLab, 3rd ed., David Hill, David Zitarelli, Prentice-Hall, 2003.

(4) Chuck Weibel, Lew Hirsch, http://math.rutgers.edu/~weibel/studies.html, Rutgers University, July 2002.