PRECALCULUS: CONCEPTS IN CONTEXT

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Precalculus at Eastern Connecticut State University includes a laboratory/project component. This paper presents examples of projects, explorations and labs from *Precalculus: Concepts in Context* [1] that have been assigned in this course. *Precalculus: Concepts in Context* is a combined text/projects-explorations-laboratory manual. This text uses a modeling approach – mathematics appears in context, and contextual problems motivate development of the mathematics needed to solve them. Projects and explorations consist of sequences of questions on single topics. Labs are designed to be collaborative ventures that culminate with a written report submitted by the group. Examples of student work are included in the discussion that follows.

The project Jail Time is assigned after students have a rudimentary background in linear and exponential functions. In this project, students develop six mathematical models describing California's prison population over time. Initially, students are told that the prison population in California in 1980 was 24,570 and that it grew to 29,200 by 1981. From this information students create two models describing California's prison population, $y_1(t)$ and $y_2(t)$, where t represents years since 1980:

- a linear model based on the assumption of a constant annual rise in prison population, $y_1(t) = 24{,}570 + 4630t$.
- an exponential model based on the assumption of a constant yearly percentage increase in prison population, $y_2(t) = 24570(1.1884)^t$.

Students test their models against the data in Figure 1 by graphing $y_1(t)$, $y_2(t)$, and a scatterplot of the prison data (See Figure 2.).

Year	1980	1981	1982	1983	1984	1985
Population	24,570	29,200	34,640	39,370	43,330	50,110
Year	1986	1987	1988	1989	1990	
Population	59,484	66,975	76,170	87,300	97,310	

Figure 1. California prison population data from 1980 to 1990.

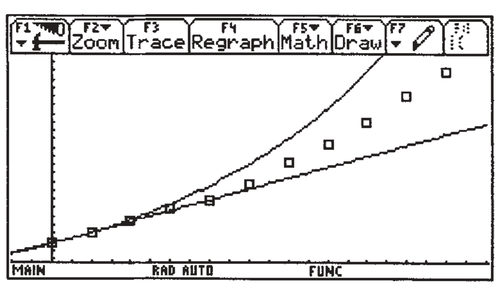

Figure 2. Graphs of models $y_1(t)$ and $y_2(t)$.

Based on these graphs, students note that the growth rate in the linear model, $y_1(t)$, does not keep up with the actual growth in California's prison population and that the exponential model, $y_2(t)$, grows too quickly. Next, students try to improve each of these models by adjusting the slope in the linear model and the growth factor in the exponential model. While student models differ, most students end up with modified models close to those below:

- modified linear model, $L(t) = 24{,}570 + 5700t$
- modified exponential model, $E(t) = 24{,}570(1.15)^t$.

Graphs of the modified functions appear in Figure 3.

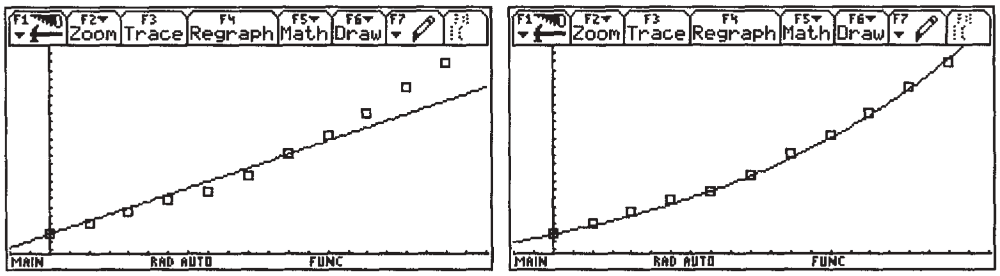

Figure 3: Graphs of modified models $L(t)$ (left) and $E(t)$ (right).

For the final two models, students use their calculator's regression capabilities to fit a linear and an exponential model to these data. The concluding question of this project asks students to use each of their models to predict the prison population in 2006 and then to compare their predictions with one made by the California Department of Corrections.

The Jail Time project strengthens students understanding of the assumptions underlying linear and exponential models. In addition, from adjusting the slope and growth factor of their linear and exponential models, students gain a clearer concept of how these parameters affect the graphs of their models. Students also learn that in mathematical modeling it may be necessary to start with a preliminary model and then to refine it.

In Jail Time, model refinement involved modifying the parameters governing growth. In the next example, from the Graph Trek lab, students discover how to modify a function, $f(x)$, by applying one of the following transformations to its input or output:

- add a constant, $f(x+c)$ and $f(x)+c$
- change the sign, $f(-x)$ and $-f(x)$
- multiply by a constant, $f(c \cdot x)$ and $c \cdot f(x)$.

Before exploring this problem through graphing, students first practice expressing the results of each of these transformations algebraically. Working with function notation is often a struggle for students. (How many of your students think that $f(x)$ is *f times x*?) As a preliminary exercise to this lab, students are given several functions (for example, $f(x) = 3x^2 - 2x + 5$) and then are asked to evaluate $f(x)+1$, $f(x+1)$, $-f(x)$, $f(-x)$, $f(2x)$, and $2f(x)$. In order to successfully explore the effect of each transformation, students also need a toolbox of functions. We suggest the toolbox include

$y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = |x|$, $y = a^x$, and $y = \sin(x)$. Using a graphing calculator, students must be able to graph each of these functions prior to this lab.

With the preliminaries out of the way, students use their graphing calculators to explore the effect that each transformation has on a given function. Students must choose functions wisely. In the discussion below, we present several examples of poor function choices made by our students.

For the first set of transformations, $f(x+c)$ or $f(x)+c$, students are often surprised that the transformation $f(x+c)$ for $c > 0$ shifts the graph of $f(x)$ c-units to the left. Figuring out why this occurs gives students added insight into this transformation. On occasion we have had students choose linear functions for $f(x)$. This choice makes it impossible to distinguish between horizontal and vertical shifts since every vertical shift of a linear function is equivalent to some horizontal shift.

Analyzing student work related to the transformations $f(-x)$ and $-f(x)$ can uncover basic misconceptions. Since the function $f(x) = x^2$ worked so well for the first set of transformations, many students continue using this function when investigating the effect of changing the sign of the input or output of a function. There are two problems related to using the function $f(x) = x^2$ to investigate the graph of $f(-x)$. Students who correctly enter $f(-x)$ into their calculator sometimes think they have made a mistake when their graph looks exactly the same as the graph of $f(x)$. Still others enter the function as $-x^2$ not understanding how their calculator deals with the order of operations involved in this problem. In this case, students erroneously conclude that changing the sign of the independent variable causes a reflection about the x-axis. One solution to both of these problems is to remind students to check their results using several different functions. In particular, suggest that groups try applying both of these transformations to $f(x) = \sqrt{x}$. The typical student response to this suggestion is that $\sqrt{-x}$ is impossible. However, when they graph this function, they discover both the domain of the function $y = \sqrt{-x}$ and the effect of changing the sign of the input. Student explanations of how changing the sign of the input or output affects a function's graph can be enlightening. Below is an excerpt from one student lab report. Clearly, this group of students has come to a very clear understanding about reflections.

> On a graph, each number is a certain number of units from an axis. . . . On a graph, the only difference between the opposite of a number and the number itself is that the opposite is on the other side of the axis. Therefore, they are in fact the same number of units. For instance, 2 is 2 units from 0. It's opposite , -2, is also 2 units from 0, except that it will be on the other side of 0. Therefore, anything above the x-axis will be that many units below it and vice versa, creating mirror images through negatives.
>
> If the rule of order of operations is followed, then the position of the negative sign does make a difference. For instance, if a negative is outside of the square

root, it is part of the output. However, if it is inside the square root, then it is part of the input. This is true for other functions, not just square roots. Being part of the input or output makes a very distinct difference. If the negative is part of the output, the x-axis is the axis of reflection because the output is of what the opposite is found. If the negative is part of the input, then the y-axis is the axis of reflection.

For determining the affect of multiplying the input or output by a constant, $f(c \cdot x)$ and $c \cdot f(x)$, many students again return to the function $f(x) = x^2$. However, using this function, every vertical stretch is equivalent to some horizontal compression. The sine function, however, is extremely helpful in this situation. Some students also discover that quadratic functions such as $f(x) = 3x^2 - 2x + 5$ also provide useful information.

The understanding of transformations learned in the Graph Trek lab is particularly useful when students get to trigonometric functions. Once the sine and cosine functions have been defined, students can jump into the Copycats project. This project asks students to explore how the parameters A, B, C, and D affect the graphs of $f(x) = A\sin[B(x - C)] + D$ and $f(x) = A\cos[B(x - C)] + D$. We have noticed over the years that our students often equate B and the period. During the course of this project, students discover the inverse relationship between B and the period, namely that the period equals $2\pi/B$. Because students discover this formula, they tend to remember it and have an easier time applying it.

After the Graph Trek lab and the Copycat project, students are ready for the Modeling Moonlight lab. Figure 4 contains data on the illuminated fraction of the moon's surface visible from the earth. The year 1999 was a special year since there were two blue moons (the second full moon in a single month) during the year.

Date	Day	Illuminated Fraction
1/1	1	0.99
1/7	7	0.73
1/13	13	0.19
1/19	19	0.03
1/25	25	0.55
1/31	31	1.00
2/6	37	0.71
2/12	43	0.17
2/18	49	0.05
2/24	55	0.62
3/2	61	1.00
3/8	67	0.70
3/14	73	0.15
3/20	79	0.08
3/26	85	0.69

Figure 4. Fraction of the moon illuminated at midnight (E.S.T.) by date in 1999.

First students make a scatterplot of the illuminated fraction versus the day. Most students use ZoomData to automatically select a viewing window. In this window it is difficult to see the pattern (see Figure 4, left-hand scatterplot). However, after widening the (ymin, ymax)-interval, the periodic nature of these data becomes apparent (see Figure 4, right-hand scatterplot). From this experience students learn the importance of controlling what they want from their calculator rather than blindly trusting the automatic settings.

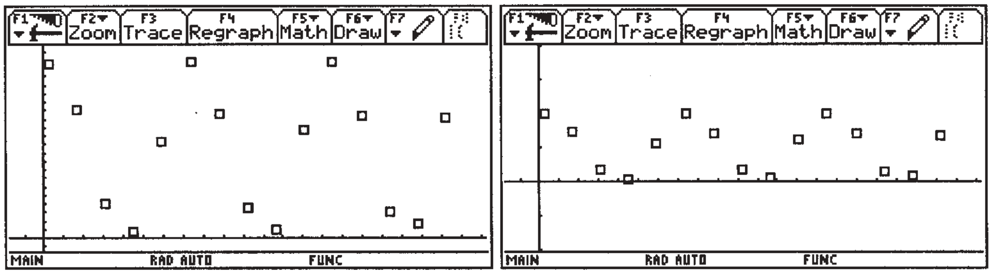

Figure 5. Scatterplots of the moonlight data using different window settings.

Since the pattern in the right-hand scatterplot resembles a cosine wave, students work to find a model of the form $y = A\cos(B(t - C)) + D$ that closely follows the pattern of these data. In this lab, students do not use their calculator's regression capabilities to fit a sinusoidal function to these data. To do so would ignore the fact that the moon oscillates between new moon (totally dark) and full moon (totally illuminated), and hence the values for A and D should be 0.5. Most students notice that the moon is full on days 31 and 61 and make $B = 2\pi/30$. Finally, students adjust the horizontal shift based on the additional information that the moon was full on January 2. Once students have a model, they can use it to approximate the rate at which the illuminated fraction changes over time. They can also determine how many blue moons occurred in 1999.

Including projects, explorations, and labs in Precalculus deepens student understanding of important mathematical concepts. Even though students often complain about the work involved in writing lab reports, they frequently cite these activities as valuable learning experiences in their course evaluations. Two sample comments follow.

- Writing lab reports . . . helped me not only learn appropriate and correct mathematical language, it forced me to confront areas of understanding about which I was unsure. Writing about larger concepts and how math is applied made me clarify connections on an entirely different level.
- I liked the AIDS, SAD, and Turtles labs because they put math into the context of situations that were not mathematical. . . . These labs forced me to understand all the information (graphs, equations, derived functions, etc.), but then it was another, separate process to tie in each bit of info to explain its relevance.

References

[1] Moran, J., Davis, M., and Murphy, M. *Precalculus: Concepts in Context 2e* Brooks/Cole Publishing Company, 2004.