MATHEMATICAL MODELING: PREDICTING THE SIZE OF THE TERROR BIRD WITH PROPORTIONALITY AND GEOMETRIC SIMILARITY

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Background on the Terror Bird

The Terror Birds were giant, flightless, predatory birds. Their ancestors lived in South America for almost 30 million years before the Interchange. Recently in Florida, there have been discovered the fossils of one Terror Bird, Titanis Walleri, which successfully made the long walk from the grasslands of South America to Florida by way of the Gulf Coastal Plain. Titanis was a fleet hunter who would lie in ambush and attack from the tall grasses. These birds killed with their beaks. After pinning down their prey with a 4 to 5 inch long inner toe claw, they could shred their prey with their massive beak. Another unique physical characteristic of these birds was that they had arms, not wings. The arms of Titanis were most like those of a bipedal dinosaur but were even more robust and powerful than those of any Velociraptor.

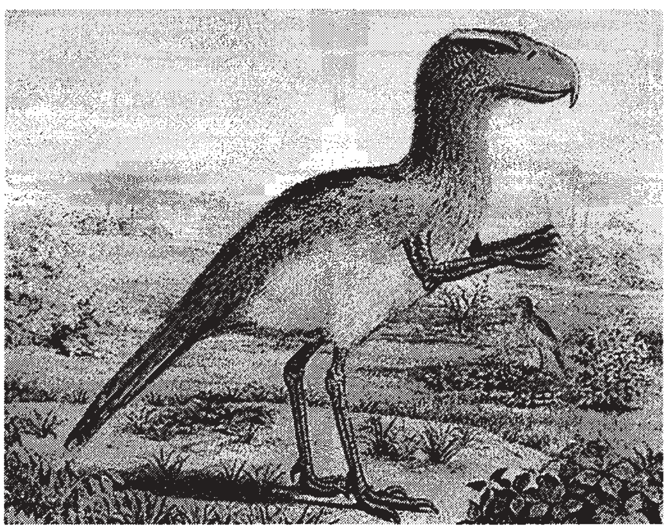

Figure 1. Artist's rendering of the Titanis Walleri

While bone sizes give good indication of the Terror Bird's size, getting reliable value for its weight is problematic: body weight does not fossilize. However, we still must be able to infer body weight from bone size using "comparative anatomy" coupled with mathematics.

Using Mathematical Modeling to Infer the Body Weight

Since the body weight of the Terror Bird cannot be directly measured from its fossil bones, scientists need a way to infer the body weight from a quantity, which *can* be measured from fossils. The goal is to find a measurable feature of the fossil bones that is somehow related to body weight. To find the needed relationship, biologists study many modern species of birds with features similar to the Terror Bird's. The basic idea is to observe how various body parts "scale up" in size as the size of the animals increase. More precisely, comparisons of adult specimens from each of several related species over an increasing range of sizes reveal that relative dimensions of various body parts increase at different rates. Thus, a lion will not only have a thicker leg bone than a house cat, it will have a *proportionately* thicker leg bone than the house cat. This relative scaling of sizes is termed an "allometric relationship."

The mathematical form of an allometric function is: $Y=C\,x^{p}$, (1)

Where Y is the dependent variable, x is the independent variable, C is a positive constant, and p is a positive power. Many simple geometric measurements are related by such allometric equations. But where did this allometric function originate.

Explicative Models: Proportionality and Geometric Similarity Arguments

We begin with a formal definition of *proportionality*. Two positive quantities x and y are said to be proportional (to each other) if one quantity is a constant positive multiple of the other. This definition implies that, $y=kx$ for some positive constant k. We write $y \propto x$ to indicate that the quantity y is proportional to the quantity x. Thus, $y \propto x$ if and only if $y = kx$ for some $k > 0$.

Let's visualize the geometric interpretation of proportionality. In Figure 2, we plot the geometric interpretation of $y \propto x$. The constant k represents the slope of the straight line that transverses the origin. The slope is defined to be the $\frac{rise}{run} = \frac{\Delta y}{\Delta x}$. This yields a line through the origin with positive slope, k (see figure 2).

Geometric similarity is a concept related to proportionality and is very useful in simplifying this portion of the mathematical modeling process. The definition is as follows: two objects are called geometrically similar if there is a one-to-one correspondence between points of the object such that the ratio of distances between corresponding points is constant for all possible pairs of points. You can think of geometrically similar objects as scale models of one another.

We will begin with two-dimensional objects. Consider the rectangles in Figure 3. Let l denote the distance from A to B in rectangle 1 and let l' denote the distance from A' to B' in rectangle 2. All corresponding points in the figures and their associated distances are also mark. For rectangles that are scale models, it must be true that

$$\frac{l}{l'} = \frac{w}{w'} = k \text{ for a constant } k > 0.$$

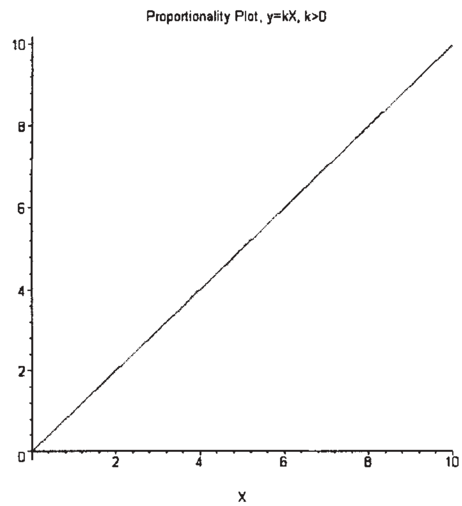

Figure 2. Proportionality Argument

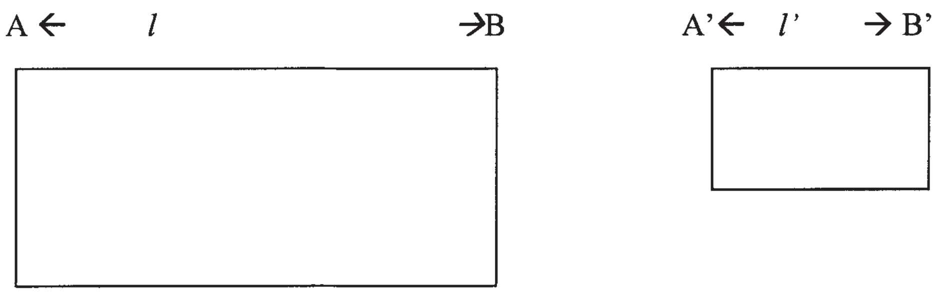

Figure 3. Scale Models

The definition is as follows: two objects are called geometrically similar if there is a one-to-one correspondence between points of the object such that the ratio of distances between corresponding points is constant for all possible pairs of points. You can think of geometrically similar objects as *scale models* of one another. This allows us to make the comparison of the Terror Bird to similar species in which we have data.

Modeling the Terror Bird

Problem Identification: Predict the weight of the terror bird as a function of the circumference of its femur.

Assumptions and Variables: We will assume that the terror birds are *geometrically similar* to other birds of today or prehistoric dinosaurs from the past. With the assumption of geometrically similarity, we can state that the volume of the bird is proportional to any characteristic dimension cubed and that the area of the bird is proportional to any characteristic dimension squared: $V \propto l^3$ and $S \propto l^2$

If we assume a constant weight density, then a volume displaces an amount equal to its weight, $V \propto W$. Thus, $V \propto W \propto l^3$. We will let the characteristic dimension be the circumference of the femur. The femur was chosen because it supports the body weight. Thus, **$W = kl^3$, $k > 0$**, where k is the positive slope constant (2)
Note that this form matches the allometric equation (1).

Modeling with Proportionality-Terror Bird Example
Modeling with proportionality and slopes. Here we want to model the weight of the terror bird as a function of the circumference of the femur. We have already built the notional model that W is proportional to any length cubed. Now, we need to check this model

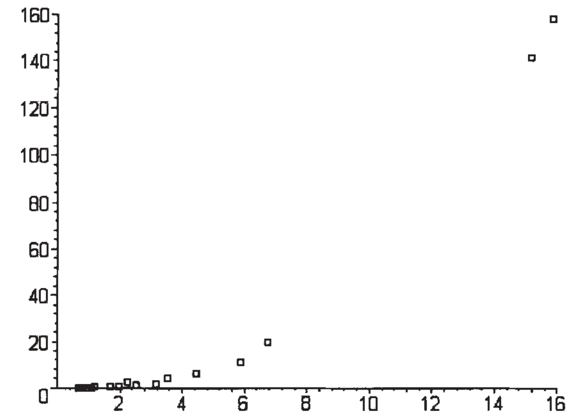

Figure 4. The scatterplot showing a curved trend

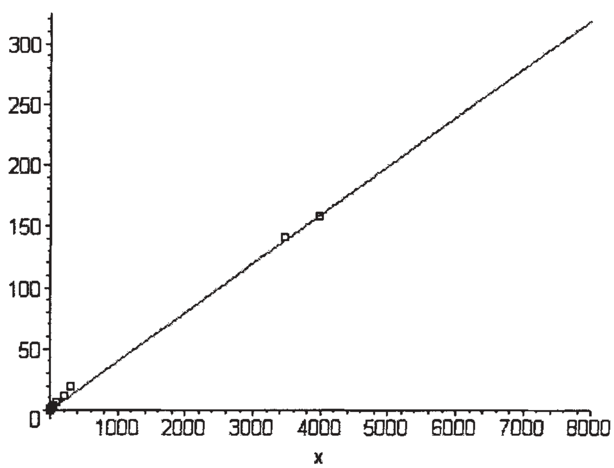

Figure 5. Proportionality Plot, a line Y = 0.039892 X

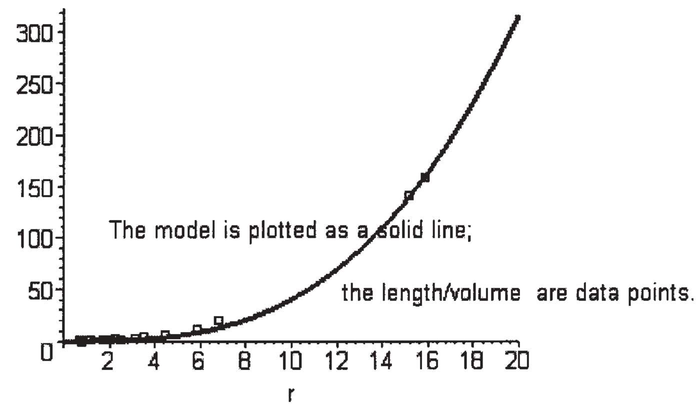

Figure 6. Regression Fit, $y= 0.04032417677\ x^3$.

The model's residuals show some trends so we might conclude that more work can be done in modeling. If we accept the model, then we can predict the weight of the Terror Bird, whose femur was 21 cm. The model used predicts the Terror Bird weigh 373.44 kg.

We might try the dinosaur data in our model or another mathematical modeling technique to obtain better estimates.

SUMMARY

Which of our mathematical models best predicts the weight of this Terror Bird?

Model	*Dinosaur Data*	*Bird Data*
Proportionality	244.12 kg	373.44 kg
ln-ln Empirical	346.96 kg	303.29 kg
Interpolating Polynomial	363.44 kg (8^{th} Order Polynomial)	N/A
Low Order Polynomial	288.7336 kg (quadratic)	None
Cubic Splines	349.029 kg	None

References

Fox, William P., Richard West, Jack Robertson, Robert Chandler, Peter M. Jarvis, and Robert Viau, "Determining the Size of the Terror Bird", ILAP, Project Intermath, 2003.