From Pythagoras To Pick Using Technology: Excerpts From A Mathematics Teacher Education Course

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Introduction

In what follows, a triple of integers (x,y,z) satisfying the equation

$$x^2-2xy\cos\gamma+y^2=z^2 \quad (1)$$

will be referred to as a γ-triple. From a geometrical viewpoint, equation (1) represents the Law of Cosines for a triangle with sides *x*, *y*, *z*. The family of Pythagorean triples can be considered as the special case of γ-triples where γ=90°. Just as every Pythagorean triple can be associated with a right triangle, every γ-triple has a γ-triangle associated with it as well. Similarly, just as the set of Pythagorean triangles with integral side lengths can be thought of as a family of triangles, every γ has an associated family of γ-triangles. In a paper presented to the 15th ICTCM [1], the authors developed spreadsheet-based computational tools that systematically generate γ-triples and the associated triangles for different values of the angular parameter γ. It has been shown that such tools allow for the meaningful introduction of several concepts of elementary number theory to preservice teachers of secondary mathematics through what can be referred to as the tool kit approach [2].

This paper is a continuation of the authors' previous work and it develops new tools that enable a variety of inquiries into analytic and geometric properties of γ-triples (Pythagorean triples in particular). Analytical inquiries of this paper deal with the frequency of numbers with special properties, including polygonal and palindromic numbers, among different families of γ-triples. The authors argue that experimental results and attempts to generalize these results provide pre-teachers of secondary mathematics with a research-like experience needed for successful teaching of the standards-based curricula.

The geometrical inquiries discussed in this paper deal with the use of a spreadsheet as a computational geoboard. First, such a geoboard can be used for the development of a computerized version of Pick's theorem [3] that calculates areas of arbitrary n-gons ($n \le 6$) with vertices on lattice points. Experimentation in a guided-discovery learning environment through the tool kit approach can lead to one's conjecturing of this less known yet pedagogically useful and mathematically simple theorem. The above-mentioned constraint on the number of edges does not limit these activities allowing one to improve technology-enabled pedagogy suggested elsewhere [4] by reducing it to a single computational medium. Furthermore, secondary pre-teachers can be engaged in explorations with a classic geometric flavor; for example, locating cosine triples corresponding to triangles with equal areas or equal perimeters [5]. This paper suggests that such use of technology in collegiate mathematics education courses has a potential to enrich mathematical experience of preservice teachers and can result in their appreciation of the power of technology and usefulness of mathematical concepts as modeling tools in computational applications.

Polygonal numbers and γ-triples

As mentioned elsewhere [1], integer solutions to equation (1) can be generated through the following formulas

$$x=q(m^2+2mn),\ y=q[2mn+2(1-p/q)n^2],\ z=q[m^2+2(1-p/q)(mn+n^2)] \qquad (2)$$

where (p, q) and (m, n) are pairs of relatively prime integers, $m>n$ and $p/q=\cos\gamma$. Using formulas (2) one can systematically generate integral (though not necessarily primitive) γ-triples that can then be explored in terms of possessing special properties.

One of the most ancient mathematical concepts is that of a polygonal number. Having historical and aesthetic appeal, special cases of polygonal numbers, namely triangular numbers, have been integrated into the pre-college mathematics curriculum starting in the early grades. These numbers can be used as tools in many grade-appropriate problem-solving situations such as finding the number of different sized rectangles on a square checkerboard problem [6] which, in particular, is equivalent to finding the sums of all numbers in the corresponding multiplication table.

An interesting exploration is to locate polygonal (triangular, square, pentagonal) numbers among Pythagorean triples generated in two different ways: first using a set of ancient formulas known to Euclid and then formulas (2). For example, there are, respectively, 28 and 47 appearances of triangular numbers among the triples generated in the two ways within the range [2, 20] of generator *m*. In much the same vein, these (and other) numbers can be located among other γ-triples within the same range of generator *m* and then results can be compared.

Palindromic numbers generated from $\tilde{\gamma}$triples

There are many interesting number theoretic activities dealing with the properties of γ-triples formulated in terms of their digits that can be significantly enhanced through the use of spreadsheets. The search for palindromic $\tilde{\gamma}$triples is an example of such an activity. (A palindrome is defined as a whole number that reads the same backward and forward.) One may argue that the visual identification of palindromes among the strings of $\tilde{\gamma}$triples is an easy task and it does not require the use of a computer, however the incorporation of computational tools makes the activity more mathematical (i.e., focus on concepts), less tedious, and more powerful (i.e., the program is efficient). This raises the following question: How can the mathematical topics of $\tilde{\gamma}$triples and palindromes be integrated meaningfully using a computer spreadsheet program? With this in mind, spreadsheets can extend inquiry into the palindromic properties of $\bar{\gamma}$triples by allowing the search for palindromes among whole number multiples of the elements of these $\bar{\gamma}$triples.

In the spirit of ideas presented in [7] concerning the exploration of parametrically generated palindromes through a whole number multiplier, a $\bar{\gamma}$triple (x, y, z) will be referred to below as the *k*-palindromic of rank one/two/three if exactly one/two/three of the products *kx*, *ky*, *kz* is a palindrome. For example, the 60°-triple (77, 32, 67) is a 1-palindromic triple of rank one. Using the spreadsheet-enabled identification of palindromes [8], preservice teachers can discover that the 120°-triples (65, 88, 133), (69, 91, 139), and (352, 123, 427) are 66-palidromic triples of the rank one, two, and three respectively. For example, the triple (66·69, 66·91, 66·139)=(4554, 6006, 9174) has exactly two palindromes. Furthermore, they can construct a numeric table and a graph that represent a numerical function P(k,γ, m) – the total number of *k*-palindromic γ-triples of all three ranks that exist for all generators not greater that *m*. The teachers then can make and test conjectures about P(k,γ, m) and the palindromic properties of $\bar{\gamma}$triples more generally using the spreadsheet as an exploratory tool. Clearly, the nature of the mathematical activity is significantly altered through such use of a spreadsheet. The authors argue that the meaningful use of various spreadsheet-based computational tools is a valuable mathematical experience for preservice teachers who are required to incorporate technology in their teaching of mathematics.

The Electronic Geoboard

In an earlier work by the authors [1], a spreadsheet was used to generate a Bride's Chair representation of Pythagorean triples. This representation (Figure 1) is dynamic in that it changes with each new input or with each new Pythagorean triple generated. Since its vertices lie on lattice points, the area of the square built on the hypotenuse – the tilted square – could be calculated using Pick's formula. This application of Pick's formula prompted the construction of a more general spreadsheet-based computational geoboard. The development of such a geoboard will be described in this section.

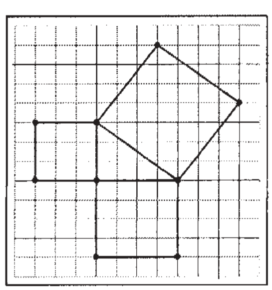

Figure 1.The Bride's Chair

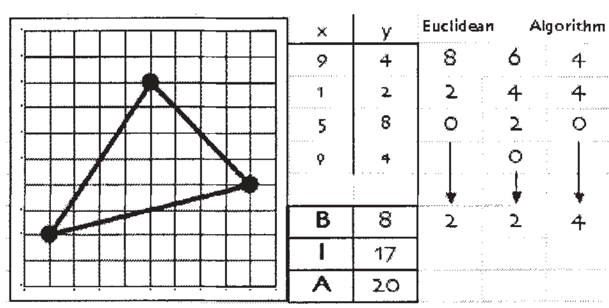

Figure 2. Electronic geoboard: First iteration.

The first iteration of the computational geoboard works for acute, right, and many, but not all, obtuse triangles. The environment (Figure 2) has the following components: (1) a (graphical) representation of 10x10 geoboard for triangles, (2) an (x,y)-table of the triangle's vertices, (3) the incorporation of the Euclidean algorithm in the count of boundary pegs (B), and (4) formulas which calculate the number of interior points (I), and the area (A), of the represented polygon.

The number of lattice points (LP) found on each edge of a polygon can be calculated through the formula LP=GCD(rise, run)+1 where the rise and run are those of the edge itself [3]. The Euclidean algorithm calculates the greatest common divisor of the rise and run of each edge. The sum of the number of lattice points diminished by three is the total number of boundary points, B, for the polygon.

Once B is known the number of lattice points on the interior of the triangle, I, can be calculated. Figure 3 shows (a) the interior triangular region drawn with a continuous line, (b) the rectangular region that circumscribes this triangle, and (c) the three right triangular regions built between the circumscribed rectangle and the three sides of the interior triangle. The method of computing the number of interior pegs results in throwing out the number of lattice points located in each of these three right-triangular regions as calculated as [(rise +1)(run +1) – LP]/2 on each region. In Figure 3, these "extra" lattice points are shown as black squares, the lattice points that are counted as I are shown in white squares.

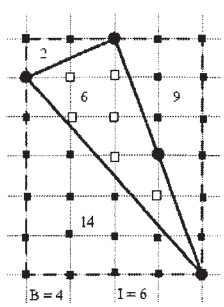

Figure 3. In/exterior pegs.

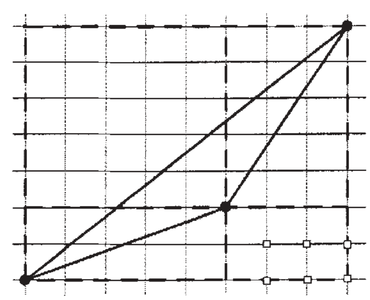

Figure 4. Problematic Triangle.

This "first iteration" of the electronic geoboard works for many triangles, but it breaks down when the triangle is obtuse and one of its vertices lies in the interior of the larger circumscribed rectangle as shown in Figure 4. In this case, the three right triangular regions will not cover all of the lattice points in the circumscribed rectangular region. In the case of Figure 4, the 6 lattice points covered by white squares are overcounted by the above methods. However, to correct for this problem, the "middle" vertex (the vertex with non-maximal valued coordinates) can be used to create four smaller rectangular sub-regions within the larger circumscribed rectangle as shown in Figure 4. The program then recognizes and adjusts for the number of lattice points located in the (smaller) rectangular region (i.e. the region that intersects the interior triangle at only one point and not an edge of the triangle.)

Moving from triangles to quadrilaterals and, ultimately, *n*-gons, causes some problems related to convexity and concavity. However, assuming a quadrilateral is created, an obvious way to extend the triangular version of the electronic geoboard to *n*-gons is by splitting the *n*-gon into smaller triangular sub-regions and then iteratively applying the counting methods developed above. If the polygon is convex then an extension from triangles to more general polygons is unproblematic, if the polygon is concave then some of the diagonals may leave the n-gon causing this approach to fail. One solution to this is to find the center of gravity for the quadrilateral, and then using this point to construct sub-triangles. Such an approach works for all 5-gons and lower and many hexagons as well.

To conclude note that using a spreadsheet as a computational geoboard preservice teachers can generate two-dimensional table of areas of lattice polygons from which the dependence of A on B and I (that is, Pick's formula) can be conjectured. There are many activities and explorations for preservice teachers that relate to the ideas that were discussed here. Among other things, preservice teachers can generate a table of values from the electronic geoboard to try to develop Pick's Formula or they could construct a simple electronic geoboard themselves.

References

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