

A MODELING EXTRAVAGANZA

Robert E. Kowalczyk and Adam O. Hausknecht
University of Massachusetts Dartmouth
Mathematics Department, 285 Old Westport Road, N. Dartmouth, MA 02747-2300
rkowalczyk@umassd.edu and ahausknecht@umassd.edu

Realistic models are the heart and soul of any mathematics course. In most first- and second-year mathematics courses, there are a multitude of topics where real-world models can be integrated into the curriculum. With modern-day technology, computer software, and digital cameras, it is possible to develop models that stimulate student interest in the mathematics being studied. Some example experiments and models that we have developed and have used to motivate our students are given below.

Modeling the Deflection of a Springboard — Linear Data

A two-foot length of springboard (lattice board — $7/8$ inch wide and $1/4$ inch thick) was fastened at one end and a number of washers were hung from the free end (see Figure 1). The purpose of this experiment is to find the functional relationship between the vertical displacement of the free end and the number of washers hanging from the free end. In this experiment, washers were added two (or three) at a time until a total of 12 (or 24) washers was reached. Students measure the amount of vertical displacement of the board after each set of washers is added and plot the resulting data. Students observe that the plotted data appears to be linear and they fit a least squares line to the data (see Figure 2). Students can record the data by using the yardstick placed directly behind the free end of the springboard or you can make a composite picture of the sequence of deflected springboards and have all the students in the class participate in the measuring of the data. The data can be measured from the picture using a ruler or the picture can be imported into TEMATH¹ and students can use TEMATH's tools to measure, plot, and fit the data. This experiment was repeated for a $1\ 3/4$ inch wide lattice board. Students can compare the slopes of the two fits and interpret the meaning of the two different slope values (see Figure 2). Rather than using an impersonal table of numbers from a book, this experiment actively involves students in generating linear data. From participating in this experiment, students develop a better, more visceral understanding for the concepts of slope and rate of change since they see how these concepts are used in the context of a physical system.

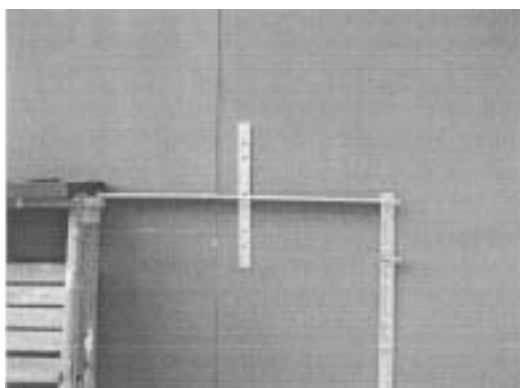


Figure 1 Springboard Set-up

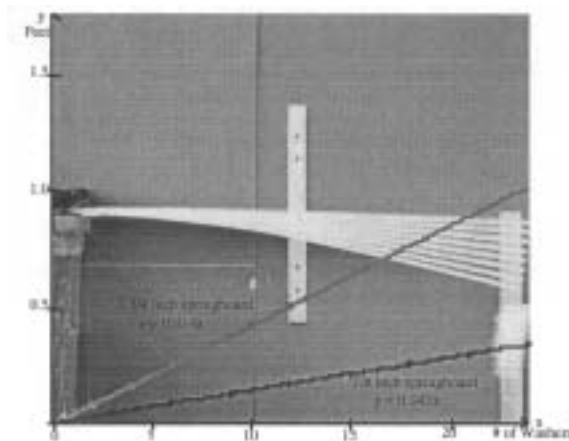


Figure 2 Modeling a Springboard

Capillary Action — Hyperbolic Data

We obtained the idea for this experiment from a project in Larson, Hostetler, and Edwards' *Calculus*² book. We purchased two plates of glass measuring 9" x 13". Using clear tape, we fastened together the left edges of the glass plates and placed a large paper clip between the right edges to form a narrow wedge-shaped channel between the glass plates. Finally, we clamped the plates together. We filled a glass rectangular dish with colored water and hung the glass plates from a frame we built so that the bottom edges of the glass plates were submerged in the water (see Figure 3). Within minutes, capillary action took place and water moved upward into the narrow channel forming a hyperbolic arc. It can be shown for this experiment that the height of the water in the channel is given by

$$h = \frac{1.000124T}{\rho g y} \approx \frac{T}{\rho g y}$$

where T = surface tension of the water (0.0728 N/m at 20° C), ρ = density of water (1000 kg/m³), g = acceleration due to gravity (9.80665 m/s²), and y = one half the distance between the glass plates within the narrow channel. We used a digital camera to take a picture of the hyperbolic arc, imported it into TEMATH, plotted the hyperbolic model on top of the picture and were amazed by the excellent fit (see Figure 4). With this experiment, students have the chance to experience an interesting physical phenomenon firsthand. The surprising accuracy of the fit increases our student's appreciation for mathematical modeling and provides a great motivational demonstration for our first-year calculus classes.

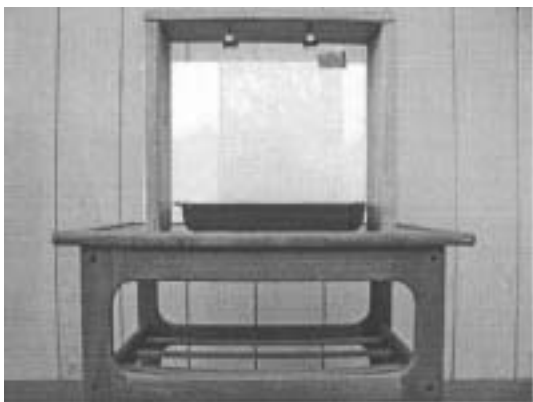


Figure 3 Capillary Action Set-up

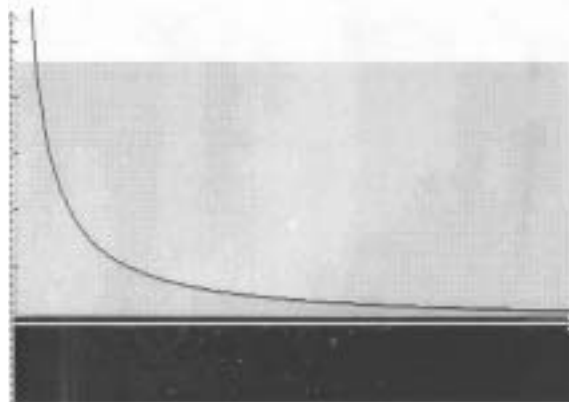


Figure 4 Capillary Action Hyperbolic Fit

The Physical Pendulum — Periodic Data

We constructed a physical pendulum from a two-foot piece of copper tubing purchased from a local hardware store. A hole was drilled through the top portion of the tubing and it was then nailed to a wall (see Figure 5). We set the pendulum into motion and took a video of the swinging pendulum for various initial amplitudes. The video was imported into a computer and converted into a sequence of frames. These frames were then imported into TEMATH and the angle of the pendulum from the vertical was measured for each frame (at evenly spaced values of $t = 1/15$ sec.). We chose the first frame to correspond to the pendulum's position at the highest point in its swing. At this point, the velocity of the pendulum is zero giving us an initial condition for the differential equation.

Measurements were made for small, medium and large initial values of θ . The Physical or Compound Pendulum³ can be modeled by the second order differential equation $-mgd\sin(\theta) = I\theta''$, where I = the moment of inertia about the axis through the pivot, d = distance from the center of mass to the pivot, and θ = angle of the tube measured from the vertical. For a uniform tube of mass m and length L pivoted about one end, $I = mL^2/3$ and $d = L/2$. Substituting these values into the differential equation, we get $\theta'' = -\frac{3g}{2L}\sin(\theta)$. This is a nonlinear second order differential equation with no closed-form theoretical solution. If the approximation $\sin(\theta) \approx \theta$ is made for small values of θ , then the differential equation becomes $\theta'' = -\frac{3g}{2L}\theta$, an easily solvable linear second order differential equation. In this experiment, we want students to test the validity of this approximation. For small initial amplitudes of the pendulum, both models fit the data well (see Figure 5). However, for larger amplitudes, the difference between the two models is significant (see Figure 6). This periodic data can be used in a general math class to discuss the properties of periodic functions and it can be used effectively in a differential equations class when discussing linearization.

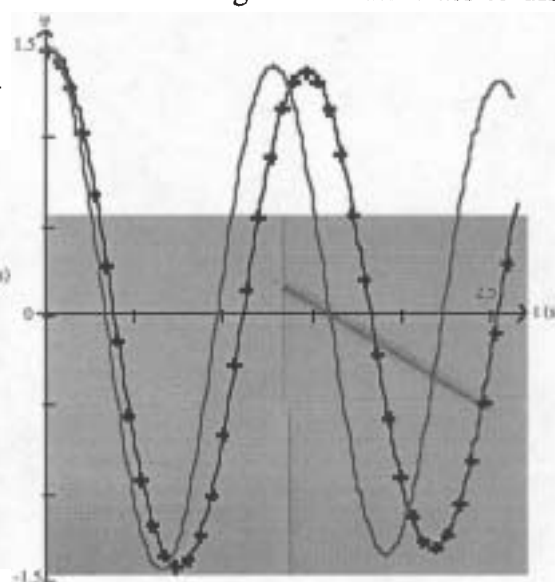
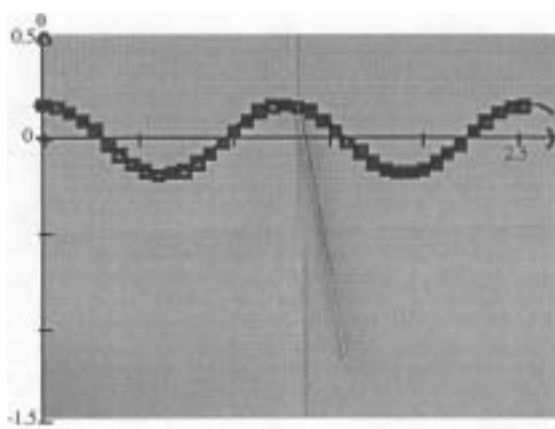


Figure 5 Pendulum — Small Amplitude Figure 6 Pendulum — Large Amplitude

Motion of a Block Sliding Down an Inclined Plane — Quadratic Data

In this experiment, we want to model the motion of a block sliding down an inclined plane and to measure the coefficient of kinetic friction of the sliding block. The sum of the forces in the direction along the inclined plane (x -direction) equals ma_x , that is, $\sum F_x = mg\sin(\theta) - \mu_k mg\cos(\theta) = ma_x$, where m = mass of the block, g = acceleration due to gravity, θ = angle of the incline, and μ_k = coefficient of kinetic friction. Solving for a_x , we get $a_x = g(\sin(\theta) - \mu_k \cos(\theta))$. The relationship between initial velocity v_0 , the velocity v_d at some distance d down the incline, and acceleration is given by $v_d^2 = v_0^2 + 2a_x d = 2a_x d$ (since $v_0 = 0$). Solving for a_x , substituting the resulting expression into $a_x = g(\sin(\theta) - \mu_k \cos(\theta))$, and finally solving for μ_k , we get $\mu_k = \frac{g\sin(\theta) - v_d^2/(2d)}{g\cos(\theta)}$.

The distance traveled along the inclined plane is modeled by the quadratic equation $s = a_x t^2 / 2 = g(\sin(\theta) - \mu_k \cos(\theta))t^2 / 2$. In order to calculate the coefficient of kinetic friction and to model the position of the block sliding down the incline, we need to measure the angle θ of the inclined plane, the velocity v_d of the block, and a distance d traveled by the block. To perform this experiment, we set up a formica covered bookshelf at an incline, placed a wooden block on the highest point of the inclined shelf and took a digital video of the block sliding down the incline. We converted the movie to a sequence of frames spaced 1/15 of a second apart, imported the frames into TEMATH, and used TEMATH's Point tool to record the position of the block in each frame. We estimated v_d by finding the difference of the last two positions of the block and dividing by 1/15. Using TEMATH's Line tool, we measured the distance d from the initial position of the block to the midpoint of the last two positions, and, we also measured the angle of the incline. For this particular experiment, we calculated μ_k to be 0.32. Figure 7 shows how well the position model fits the measured data. Students can perform this experiment for a variety of materials for both the block and the inclined board. This activity fits well in a first-year calculus course taken by engineering and physics majors.

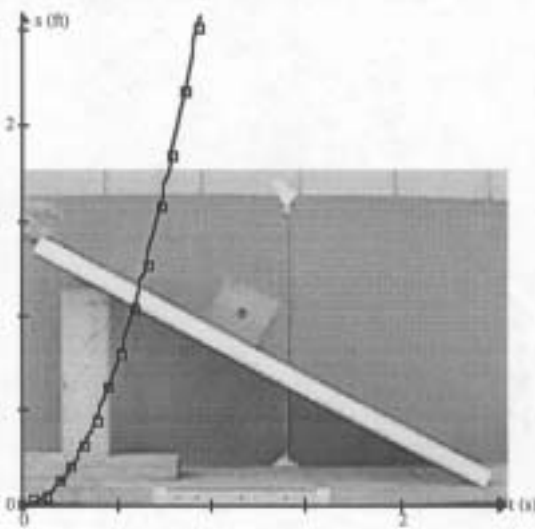


Figure 7 Modeling Kinetic Friction

Linear Programming Models — System of Inequalities

TEMATH has a tool for generating and plotting the region of feasible solutions for a system of inequalities. TEMATH also finds all corner points and displays a plot of the objective function. You can drag the plot of the objective function across the region of feasible solutions and observe its value as it is moved. Using this visual tool, students can form a conjecture as to where the objective function achieves a maximum or minimum value. You can also use TEMATH to traverse all the corner points by moving the objective function from one corner point to the next. An example linear programming problem is given below along with TEMATH's graphical output shown in Figure 8.

$$\text{Maximize } z = 5x + 8y$$

Subject to

$$0.5x + 0.5y \leq 6.5$$

$$1.0x + 2.0y \leq 22$$

$$2.0x + 1.0y \leq 20$$

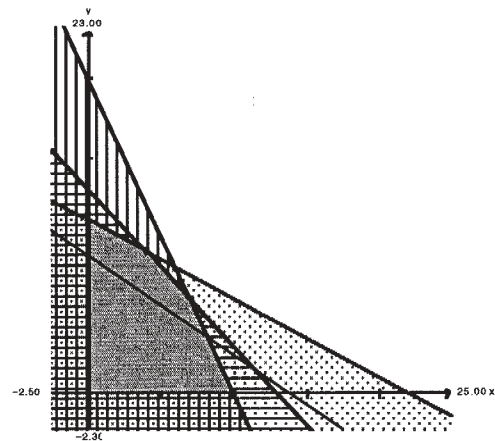


Figure 8 Linear Programming Model

Two dimensional and higher order linear programming models can be solved using TEMATH's Matrix Calculator. Once the coefficients of the inequalities and objective function have been entered, you simply click the Linear Programming command button and the solution is found. Alternatively, click the LPStep command button to find the Simplex Tableau, and then click the OK button to take a single step of the Simplex Process. Students can step through the entire Simplex Process to both check their own work and their understanding of the Simplex Method.

A Holiday Surprise — Having Fun with Parametric Equations

In second semester calculus, students learn to find the parametric equations for drawing a line, circle, ellipse, and an arc of a circle or ellipse. To give students extra practice with finding these parametric equations, it is always a fun project to have them use parametric equations to draw some kind of design or to write their name. Around a holiday like Halloween, it is fun to have them draw a face on a pumpkin. Using TEMATH, you can create a file that contains a picture of a pumpkin as a background image for a pair of coordinate axes. Using this prepared file, students enter their list of parametric equations that draw the face of a jack-o-lantern. An example is shown in Figure 9.



Figure 9 Using Parametric Equations

TEMATH (Tools for Exploring Mathematics)

TEMATH (Tools for Exploring Mathematics) is a mathematics exploration environment useful for investigating a broad range of mathematical problems. It is effective for solving problems in pre-calculus, calculus, differential equations, linear algebra, numerical analysis, and math modeling. TEMATH contains a powerful grapher, a matrix calculator, an expression calculator, a differential equation solver, a facility for handling and manipulating data, numerical mathematical tools, visual and dynamic exploration tools, and the capability for importing digital background images. TEMATH requires an Apple Macintosh computer running MAC OS 8.5-9.2.2 or Mac OS X (as a classic application), a 12" or larger monitor screen, 3 MB of free RAM, and 2MB of disk space (for TEMATH plus its support files). You can download a copy of TEMATH and its documentation, application files, and games from:

www2.umassd.edu/TEMATH

Bibliography

- [1] Robert Kowalczyk and Adam Hausknecht, *TEMATH - Tools for Exploring Mathematics Version 2.1*, 2003.
- [2] R. Larson, R. Hostetler, B. Edwards, *Calculus*, Houghton Mifflin Co., 2003.
- [3] R. A. Serway, *Principles of Physics*, Harcourt Brace College Publishing, 1994.
- [4] Robert Kowalczyk and Adam Hausknecht, *Using TEMATH in Calculus*, 2000.